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Rapidly rotating relativistic stars

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Within the last decade, significant progress has been made in modelling rotating stars in general relativity and in relating observable properties to the equation of state of matter at high density. A formalism describing rotating perfect fluids is presented and numerical models of neutron stars are briefly discussed, with emphasis on upper limits on mass and rotation. The equations governing small oscillations are reviewed, and a variational principle appropriate both to eulerian and lagrangian perturbations is obtained. This extends to relativity an eulerian principle used to find non-axisymmetric stability points for perfect fluids. A related eulerian approach has been recently used to obtain normal modes of rotating newtonian stars. The review concludes with an outline of this work and of the two types of instability that can restrict the range of neutron stars. In particular, current work shows that several kinds of effective viscosity limit the possible role of a non-axisymmetric instability driven by gravitational waves.

0. Introduction

Following the discovery of PSR 1937+21, a pulsar rotating at frequency $0.4033 \times 10^4\ \mathrm{s}^{-1}$, the search for fast pulsars has accelerated. Within the past 10 years more than 20 pulsars with periods less than 10 ms have been discovered, although only one (PSR 1957+20) has a frequency comparable with that of the first millisecond pulsar (Backer & Kulkarni 1991; Manchester *et al.* 1991). The next decade may disclose an upper limit on pulsar rotation, and there is hope that the mass of some fast pulsars may also be observed. It is not yet clear whether even the fastest of the observed pulsars are rotating rapidly, in the sense of having a rotational energy close to its maximum value. But the growing class of millisecond pulsars has dramatized the possibility that accretion-driven spin-up of old neutron stars and accretion induced collapse of white dwarfs may each lead to rapidly rotating stars.

We review here some of the work done in the past decade on the structure, stability and oscillations of rapidly rotating relativistic stars. The article is primarily concerned with work done by the authors and their collaborators, and more emphasis is given to recent work involving stellar oscillations than to work on equilibrium models, which has been reviewed elsewhere (Friedman 1990, 1991; Parker 1990). Related work by others is mentioned but is, in general, not presented in detail.

Section 1 discusses equilibrium configurations. The equations governing perfect fluids and the construction of stellar models are reviewed in §1*b*. An extensive study has been made by several groups of rapidly rotating stellar models, based on a wide range of equations of state, and this work is outlined briefly in §1*c*. Section 1*d* concludes the outline with a discussion of upper limits on mass and rotation. The maximum rotation is sensitive to the equation of state (EOS) of matter above nuclear

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density. We emphasize limitations on the EOS that would follow from an observed limit on the angular velocity of relativistic stars.

Section 2 discusses the oscillation and stability of rapidly rotating stars. In §2*a* a formalism for discussing perturbations of relativistic stars is reviewed and extended. We relate an action for the perturbation equations based on lagrangian displacements to an action for normal modes based on eulerian perturbations that was obtained in the newtonian limit by Ipser & Managan (1985) and in the relativistic Cowling approximation by Ipser & Lindblom (1992*a*). This completes the extension to relativity of the eulerian variational principle. In §2*b*, we discuss a related formalism for treating normal modes of rapidly rotating stars. The formal development is complete only in the newtonian limit, where it has been successfully used to obtain, for the first time, non-axisymmetric modes of rotating models. Finally, in §2*c*, applications to stability are discussed. The region of stable configurations is limited by an axisymmetric instability to collapse, and, for stars with sufficiently low viscosity, by a non-axisymmetric instability that restricts the maximum rotation. Recent work on dissipative mechanisms in neutron stars suggests that the non-axisymmetric instability can only limit the angular velocity of recently formed stars.

1. Equilibrium equations

(*a*) *Notation and mathematical preliminaries*

(i) *Conventions*

Gravitational units, with $G = c = 1$, will be adopted in writing the equations governing stellar structure and dynamics, while numerical properties of stellar models will be listed in c.g.s. units. We use the conventions of Misner *et al.* (1973) for the signature of the space-time metric $(-+++)$ and for signs of the curvature tensor and its contractions. Space-time indices will be Latin a–h (and may be regarded as abstract by readers familiar with the abstract index convection). Corresponding to a choice of coordinates, t, r, θ, ϕ, a vector u^a has components $u^t, \ldots, u^\phi$; and its components along an orthonormal frame, $\{\boldsymbol{e}_{(0)}, \ldots, \boldsymbol{e}_{(3)}\}$, will be written $\{u^{(0)}, \ldots, u^{(3)}\}$.

Numbers that rely on physical constants are based on the values:

$$c = 2.9979\times 10^{10}\ \text{cm s}^{-1}, \quad G = 6.670\times 10^{-8}\ \text{g}^{-1}\ \text{cm}^{-3}\ \text{s}^{-2},$$

$$\hbar = 1.0545\times 10^{-27}\ \text{g cm}^{-2}\ \text{s}^{-1},$$

$$\text{nucleon mass } m_n = 1.659\times 10^{-24}\ \text{g}, \quad M_\odot = 1.989\times 10^{33}\ \text{g} = 1.477\ \text{km}.$$

(ii) *Notation for derivatives and integrals*

The covariant derivative operator of the spacetime metric g_{ab} will be written ∇_a, and the partial derivative of a scalar f with respect to one of the coordinates, say r, will be written $\partial_r f$ or $f_{,r}$. Lie derivatives along a vector u^a will be denoted by $\mathscr{L}_u$. The Lie derivative of an arbitrary tensor $T^{a\ldots b}_{c\ldots d}$ is

$$\mathscr{L}_u T^{a\ldots b}_{c\ldots d} = u^e \nabla_e T^{a\ldots b}_{c\ldots d} - T^{e\ldots b}_{c\ldots d}\nabla_e u^a - \ldots - T^{a\ldots e}_{c\ldots d}\nabla_e u^b + T^{a\ldots b}_{e\ldots d}\nabla_c u^e + \ldots + T^{a\ldots b}_{c\ldots e}\nabla_d u^e. \tag{1.1}$$

Our notation for integrals is as follows. We denote by $\mathrm{d}\tau$ the space-time volume element. In a chart $\{x^0, x^1, x^2, x^3\}$, the notation means,

$$\mathrm{d}\tau = \sqrt{-g}\,\mathrm{d}^4x. \tag{1.2}$$

Gauss's theorem has the form

$$\int_{\Omega} \nabla_a A^a = \int_{\partial\Omega} A^a \, dS_a. \tag{1.3}$$

In a chart (u, x^1, x^2, x^3) for which S is a surface of constant u, $dS_a = \sqrt{-g}\,\nabla_a u\,d^3x$, and

$$\int_V A^a \, dS_a = \int_V A^u \sqrt{-g}\, d^3x. \tag{1.4}$$

If S is nowhere null, one can define a unit normal,

$$n_a = \nabla_a u / |\nabla_b u \nabla^b u|^{\frac{1}{2}}, \tag{1.5}$$

and write
$$dS_a = n_a \, dV, \quad \text{where} \quad dV = \sqrt{|^3g|}\, d^3x, \tag{1.6}$$

but Gauss's theorem has the form (1.3) for any 3-surface S, bounding a four-dimensional region $\mathscr{R}$, regardless of whether S is timelike, spacelike or null.

(*b*) *Perfect fluids and equilibrium configurations*

(i) *Equations governing perfect fluids*

It is widely believed that pulsars and most compact X-ray sources are neutron stars. The uncertainty in the behaviour of matter at high density, however, is underlined by the recent attention given to the Bodmer–Witten suggestion that for collections of more than a few hundred baryons, the ground state at zero pressure may be strange quark matter instead of iron (Bodmer 1971; Witten 1984). If this is true, and if it is possible for neutron matter to tunnel to quark matter in at least some neutron stars, then it appears likely that all 'neutron stars' would have to be quark stars (Madsen 1990, 1991; Friedman 1990; Caldwell & Friedman 1991). This appears inconsistent with observations of glitches (Alpar 1988) if they require the existence of a thick crust of normal matter. (A bibliography including over 50 articles on strange stars can be found in Madsen (1992).)

In neutron stars, departures from perfect fluid equilibrium due to a solid crust are of order 10^{-5}, corresponding to the maximum strain that an electromagnetic lattice can support. The estimate is consistent with observations of glitches, sudden observed increases in the frequency of a pulsar, presumably in the angular velocity of a neutron star's outer crust. As the star spins down, its moment of inertia deviates slightly from that of a fluid because its solid crust allows small anisotropic stresses. It was initially thought that glitches were crustquakes, the quick spin-up due to a sudden decrease in the moment of inertia of the crust as the quake allowed it to relax to shape of a fluid equilibrium. Other explanations involve sudden unpinning of vortex tubes, producing a rapid redistribution of angular momentum from superfluid to crust; a hybrid mechanism in which a crust breaks to allow the pinned vortex tubes to move; and an onset of superfluid turbulence in the crust when the difference between the crust and superfluid angular velocity reaches a critical value. (See Epstein (1988) for references and a review.) In each of these models, however, the difference between the effective moment of inertia (angular momentum of star/angular velocity of outer crust) is in principle limited by the maximum strain that a solid crust can support. If some or all glitches are associated with crustquakes, the maximum fractional strain is approximated by the fractional displacement of the crust in a glitch, again of order 10^{-5}.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

On a submillimetre scale, superfluid neutrons and protons in the interior of a neutron star have velocity fields that are curl-free outside a set of quantized vortices. On larger scales, however, a single, averaged, velocity field u^a accurately describes a neutron star (Baym & Chandler 1983; Sonin 1987; Lindblom & Mendell 1991). Although the approximation of uniform rotation is consequently invalid on scales shorter than 1 cm, the error in computing the structure of the star on larger scales is negligible. In particular, with T^{ab} approximated by a value, $\langle T^{ab}\rangle$ averaged over several centimetres, the error in computing the metric is of order

$$\delta g_{ab} \sim (1\ \mathrm{cm}/R)^2 \sim 10^{-11}. \tag{1.7}$$

(For a time-independent geometry, the field equations, $G_{ab} = 8\pi T_{ab}$, can be written as coupled elliptic equations for g_{ab} with source T_{ab}. The error δg_{ab} satisfies a second-order elliptic equation whose source, $\delta T_{ab} = T_{ab} - \langle T_{ab}\rangle$, is rapidly varying. Then, writing $\partial^2 g \approx R^{-2} g, \partial^2\,\delta g \approx \lambda^{-2}\,\delta g$, where $\lambda \approx 1$ cm is the characteristic wavelength of δT, we recover (1.7).)

A perfect fluid is described by an energy–momentum tensor of the form,

$$T^{ab} = \epsilon u^a u^b + p q^{ab}, \tag{1.8}$$

where the 4-velocity u^a is a unit timelike vector field,

$$u^a u_a = -1, \tag{1.9}$$

and q^{ab} is the projection orthogonal to u^a,

$$q^{ab} = g^{ab} + u^a u^b. \tag{1.10}$$

The scalars ϵ and p are the energy density and pressure measured by a comoving observer (an observer with 4-velocity u^a). By projecting the equation of motion,

$$\nabla_b T^{ab} = 0, \tag{1.11}$$

orthogonal to the 4-velocity u^a, one obtains the relativistic form of Euler's equation,

$$u^b\,\nabla_b\,u_a = -(\epsilon+p)^{-1}\,q_a^b\,\nabla_b\,p; \tag{1.12}$$

and the projection along u^a expresses conservation of energy in the manner,

$$u^b\,\nabla_b\,\epsilon = -(\epsilon+p)\,\nabla_b\,u^b. \tag{1.13}$$

A two-parameter equation of state can be written in the form,

$$\epsilon = \epsilon(n,s), \quad p = p(n,s), \tag{1.14}$$

where n and s are baryon density and entropy per baryon, respectively. A one-parameter equation of state suffices to describe a neutron star for most of its history, because within days after formation, neutrino emission cools the stars to 10^{10} K $\approx$ 1 MeV. This is much smaller than the Fermi energy of the interior, in which a density greater than nuclear (0.18 f m^{-3}) implies a Fermi energy greater than $\epsilon_{\mathrm{F}}(0.18\ \mathrm{f\,m}^{-3})$ $\approx$ 60 MeV. A neutron star is in this sense cold, and because nuclear reaction times are shorter than the cooling time, one can use a zero-temperature equation of state (EOS) to describe the matter:

$$\epsilon = \epsilon(p) \quad \text{or, equivalently,} \quad \epsilon = \epsilon(n) \quad p = p(n). \tag{1.15}$$

We shall denote by ρ the baryon mass density,

$$\rho := m_{\mathrm{n}}\,n, \tag{1.16}$$

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

with m_n is the mass of a nucleon. Conservation of baryons is expressed by

$$\nabla_a(nu^a) = 0. \tag{1.17}$$

In terms of the scalars n and s, the second law takes the form

$$\mathrm{d}\epsilon = T\,\mathrm{d}s + n^{-1}(\epsilon+p)\,\mathrm{d}n. \tag{1.18}$$

The quantity,

$$h = (\epsilon+p)/n, \tag{1.19}$$

is the comoving enthalpy per baryon, $\tilde{h} = (\epsilon+p)/\rho$ is the enthalpy per unit rest mass, and

$$g := (\epsilon+p)/n - Ts \tag{1.20}$$

is the Gibbs free energy per baryon. In the newtonian limit,

$$\tilde{h} - 1 \rightarrow u + p/\rho, \tag{1.21}$$

the newtonian specific enthalpy, with u the internal energy per unit mass.

A stationary flow is described by a space-time with a timelike Killing vector, t^a, the generator of time-translations that leave the metric and the fluid variables fixed:

$$\mathcal{L}_t g_{ab} = \mathcal{L}_t u^a = \mathcal{L}_t \epsilon = \mathcal{L}_t p = 0. \tag{1.22}$$

Bernoulli's law is the newtonian conservation of enthalpy for a stationary flow, and its relativistic form is

$$\mathcal{L}_u(hu_b t^b) = 0. \tag{1.23}$$

To obtain (1.23), one uses the relation,

$$(u^a \nabla_a h)/h = (u^a \nabla_a p)/(\epsilon+p), \tag{1.24}$$

which itself follows from conservation of energy and baryon number, (1.13) and (1.17): that is, from

$$(u^a \nabla_a \epsilon)/(\epsilon+p) = -\nabla_a u^a = (u^a \nabla_a n)/n, \tag{1.25}$$

we have,

$$u^a \nabla_a \left(\frac{\epsilon+p}{n}\right) = \frac{1}{n}(u^a \nabla_a \epsilon + u^a \nabla_a p) - \frac{\epsilon+p}{n^2} u^a \nabla_a n = \frac{u^a \nabla_a p}{n}. \tag{1.26}$$

Because

$$\mathcal{L}_u u_a = u^b \nabla_b u_a + u_b \nabla_a u^b = u^b \nabla_b u_a, \tag{1.27}$$

the Euler equation, (1.12), becomes

$$\mathcal{L}_u(hu_a) = -(\nabla_a p)/n. \tag{1.28}$$

Contracting this form of the Euler equation with t^a and using (1.22), we obtain (1.23). The derivation holds for any Killing vector that Lie-derives the fluid variables, and, for an axisymmetric flow, yields conservation of a fluid element's angular momentum in the form,

$$\mathcal{L}_u(hu_b \phi^b) = 0. \tag{1.29}$$

From a mathematical perspective, introducing a conserved baryon number is merely convenient. Instead of defining the specific enthalpy by $\tilde{h} = (\epsilon+p)/\rho$, one can take as the definition

$$\tilde{h} = \exp \int_0^p \frac{\mathrm{d}p}{\epsilon(p,s)+p}. \tag{1.30}$$

Again one has (1.24), $$(u^a \nabla_a \tilde{h})/\tilde{h} = (u^a \nabla_a p)/(\epsilon + p), \tag{1.31}$$

implying the corresponding Bernoulli equation,

$$\mathscr{L}_u(\tilde{h} u_b t^b) = 0. \tag{1.32}$$

One needs additional physics, the relations, $\epsilon/\rho \to 1$ and $p/\rho \to 0$, as $p \to 0$ for fixed s, to make the identification,

$$\tilde{h} = h/m_n = (\epsilon + p)/\rho. \tag{1.33}$$

The flow of an isentropic fluid conserves circulation. If one defines a relativistic vorticity ω_{ab} by

$$\omega_{ab} = \nabla_a(hu_b) - \nabla_b(hu_a), \tag{1.34}$$

the differential conservation law is the curl of (1.28),

$$\mathscr{L}_u \omega_{ab} = 0. \tag{1.35}$$

The corresponding integral law is obtained as follows. Let c be a closed curve in the fluid, bounding a 2-surface Σ; and let c_τ be the curve obtained by moving each point of c a proper time τ along the fluid trajectory through that point. From the relation,

$$\mathscr{L}_u \omega_{ab} = \nabla_a \mathscr{L}_u(hu_b) - \nabla_b \mathscr{L}_u(hu_a), \tag{1.36}$$

we have $$0 = \int_\Sigma \mathscr{L}_u \omega_{ab}\, \mathrm{d}S^{ab} = \int_c \mathscr{L}_u(hu_a)\, \mathrm{d}l^a$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau} \int_{c_\tau} hu_a\, \mathrm{d}l^a, \tag{1.37}$$

where Stokes's theorem was used to obtain the first equality. That is, the line integral,

$$\int_{c_\tau} hu_a\, \mathrm{d}l^a = \int_{c_\tau} \frac{\epsilon + p}{n} u_a\, \mathrm{d}l^a, \tag{1.38}$$

is independent of τ, conserved by the fluid flow.

(ii) *Geometry of a rotating star*

The metric g_{ab} of a stationary axisymmetric rotating fluid has two commuting Killing vectors, ϕ^a and t^a, generating rotations and asymptotic time-translations. The symmetry of the metric means the vanishing of its Lie derivatives:

$$£_t g_{ab} = \nabla_a t_b + \nabla_b t_a = 0, \quad £_\phi g_{ab} = \nabla_a \phi_b + \nabla_b \phi_a = 0. \tag{1.39}$$

The commutator $[t, \phi]$ is again a Lie derivative. Its vanishing,

$$£_t \phi^a = 0, \tag{1.40}$$

implies the existence of a family of 2-surfaces spanned by t^a and ϕ^a and of scalars, t, ϕ, for which

$$t^a \nabla_a t = \phi^a \nabla_a \phi = 1, \quad t^a \nabla_a \phi = \phi^a \nabla_a t = 0. \tag{1.41}$$

The fluid's 4-velocity has the form,

$$u^a = u^t(t^a + \Omega \phi^a), \tag{1.42}$$

where $u^t = u^a \nabla_a t$, implying (Kundt & Trumper 1966; Carter 1969) the existence of

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

a family of 2-surfaces orthogonal to the Killing vectors. The metric, g_{ab}, can be written in terms of dot products of the Killing vectors, $t^a t_a$, $t^a \phi_a$, $\phi^a \phi_a$ and a conformal factor, $e^{2\mu}$, that characterizes the geometry of the orthogonal 2-surfaces:

$$\left.\begin{aligned} g^{tt} &= \nabla_a t \nabla^a t = -e^{-2\nu}, \\ g_{\phi\phi} &= \phi^a \phi_a = e^{2\psi}, \\ g_{t\phi} &= t^a \phi_a = -\omega\, e^{2\psi}. \end{aligned}\right\} \tag{1.43}$$

Then
$$g_{tt} = t^a t_a = -e^{2\nu} + \omega^2 e^{2\psi}, \tag{1.44}$$
and
$$ds^2 = -e^{2\nu}\, dt^2 + e^{2\psi}(d\phi - \omega\, dt)^2 + e^{2\mu}(d\varpi^2 + dz^2), \tag{1.45}$$
where ϖ and z are cylindrical coordinates labelling the 2-surfaces orthogonal to t^a and ϕ^a.

The Killing vectors have components,

$$t^i = \delta^i_t, \quad \phi^i = \delta^i_\phi \tag{1.46}$$

and the symmetry (1.39) means that the potentials ν, ψ, ω and μ depend only on ϖ and z. Because of the choice of an overall conformal factor, $e^{2\mu}$, to describe the geometry of the $\varpi - z$ surfaces, the exterior of a spherical star given by (1.45) is essentially the Schwarzschild geometry in isotropic coordinates,

$$e^\nu = \frac{1 - M/2r}{1 + M/2r}, \quad e^\psi = \varpi\left(1 + \frac{M}{2r}\right)^2, \quad e^\mu = \left(1 + \frac{M}{2r}\right)^2. \tag{1.47}$$

Asymptotically, the relations

$$e^\psi = \varpi(e^{-\nu} + O(r^{-2})), \quad e^\mu = e^{-\nu} + O(r^{-2}), \tag{1.48}$$

hold for the potentials, (1.47), and for the metric (1.45) as well, because any stationary, asymptotically flat space-time agrees with the Schwarzschild geometry to order r^{-1}. If, following Bardeen & Wagoner (1971), we write

$$\beta := \psi + \nu, \quad \zeta := \mu + \nu, \quad B := \varpi^{-1} e^\beta, \tag{1.49}$$

then, asymptotically, β (or B) deviates by $O(r^{-2})$ from its value in the isotropic Schwarzschild metric; and ζ, which vanishes for isotropic Schwarzschild, is itself of order r^{-2}.

The angular velocity $\omega \equiv -t^a \phi_a / \phi^b \phi_b$, measures the dragging of inertial frames in the sense that particles with zero angular momentum move along trajectories whose angular velocity relative to infinity is $d\phi/dt = \omega$. A natural tetrad is the frame of zero-angular-momentum-observers, with basis covectors

$$\omega^{(0)} = e^\nu\, dt, \quad \omega^{(1)} = e^\psi(d\phi - \omega\, dt), \quad \omega^{(2)} = e^\mu\, d\varpi, \quad \omega^{(3)} = e^\mu\, dz, \tag{1.50}$$

and the corresponding contravariant basis vectors are

$$e_{(0)} = e^{-\nu}(\partial_t + \omega\, \partial_\phi), \quad e_{(1)} = e^{-\psi}\, \partial_\phi, \quad e_{(2)} = e^{-\mu}\, \partial_\varpi, \quad e_{(3)} = e^{-\mu}\, \partial_z. \tag{1.51}$$

The non-zero components of the four velocity u^a along these frame vectors can be written in terms of a fluid 3-velocity v in the manner

$$u^{(0)} = 1/\surd(1 - v^2), \quad u^{(1)} = v/\surd(1 - v^2). \tag{1.52}$$

Then
$$u^t = u^a \nabla_a t = e^{-\nu}/\surd(1 - v^2), \quad u^\phi = u^a \nabla_a \phi = \Omega u^t, \tag{1.53}$$

where Ω is the angular velocity of the fluid relative to infinity (measured by an asymptotic observer with 4-velocity along the asymptotically timelike Killing vector, t^a). The 3-velocity, v, written in terms of Ω, is

$$v = e^{\psi-\nu}(\Omega-\omega). \tag{1.54}$$

Note that $2\pi e^{\psi}$ is the circumference of a circle centred about the axis of symmetry (the z-axis); that is, e^{ψ} agrees for spherical stars with $r\sin\theta$, where r and θ are the usual Schwarzschild coordinates (not the isotropic coordinates introduced above). As noted earlier, on scales larger than centimetres, we can represent neutron star matter by the energy–momentum tensor, $T^{ab} = \epsilon u^a u^b$, of a perfect fluid. The non-vanishing tetrad components are

$$T^{(0)(0)} = (\epsilon+pv^2)/(1-v^2), \quad T^{(0)(1)} = (\epsilon+p)\,v/(1-v^2), \tag{1.55}$$

$$T^{(1)(1)} = (\epsilon v^2+p)/(1-v^2), \quad T^{(2)(2)} = T^{(3)(3)} = p. \tag{1.56}$$

The four potentials are determined by four components of the field equation

$$G_{ab} = 8\pi T_{ab}, \tag{1.57}$$

whose selection is a matter of taste. Following Bardeen & Wagoner (1971), Butterworth & Ipser (1976) based their code on the four equations,

$e^{-\beta}R^{(0)(0)} = e^{-\beta+2\nu}R^{tt}$:

$$\nabla^a\nabla_a\nu - \tfrac{1}{4}e^{\beta-4\nu}\nabla^a\omega\nabla_a\omega = -8\pi e^{-}\beta[(\epsilon+p)(1+v^2)/(1-v^2)+2p]; \tag{1.58}$$

$e^{\beta-2\nu}R^{(0)(1)} = 2R^t_{\phi}$:

$$\nabla^a(e^{2\beta-4\nu}\nabla_a\omega) = -16\pi e^{\beta-2\nu}(\epsilon+p)\,v/(1-v^2); \tag{1.59}$$

$e^{-\beta}(G^{(2)(2)}+G^{(3)(3)}) = e^{\beta-2\mu}(G_{\varpi\varpi}+G_{zz}) = e^{-\beta}(R^{(0)(0)}-R^{(1)(1)})$:

$$\nabla^a\nabla_a\beta = 16\pi e^{-\beta}p; \tag{1.60}$$

and

$e^{2\mu}G^{(2)(3)} = G_{\varpi z}$:

$$\mu_{,\varpi}\beta_{,z}+\mu_{,z}\beta_{,\varpi}-\beta_{,\varpi z}-\beta_{,\varpi}\beta_{,z}-2\nu_{,\varpi}\nu_{,z}+\beta_{,z}\nu_{,\varpi}-\tfrac{1}{2}e^{2\beta-4\nu-2\mu}\omega_{,\varpi}\omega_{,z} = 0. \tag{1.61}$$

For reference, we list the equations corresponding to the three remaining non-vanishing components of G^{ab}

$-e^{-\beta}G^{(0)(0)} = -e^{-\beta+2\nu}G^{tt}$:

$$e^{\nu}\nabla^a[e^{-\nu}\nabla_a(\psi+\mu)] - \tfrac{1}{4}e^{\beta-4\nu}\nabla^a\omega\nabla_a\omega = -8\pi e^{-\beta}(\epsilon-pv^2)/(1-v^2); \tag{1.62}$$

$-e^{-\beta}R^{(1)(1)} = -e^{-3\beta+2\nu}R_{\phi\phi}$:

$$\nabla^a\nabla_a\psi + \tfrac{1}{2}e^{\beta-4\nu}\nabla^a\omega\nabla_a\omega = -8\pi e^{-\beta}(\epsilon v^2+p)/(1-v^2); \tag{1.63}$$

$e^{-\beta+2\mu}(G^{(3)(3)}-G^{(2)(2)}) = e^{-\beta}(G_{zz}-G_{\varpi\varpi})$:

$$\beta_{,\varpi\varpi}-\beta_{\mu,zz}-2(\beta_{,\varpi}\mu_{,\varpi}-\beta_{,z}\mu_{,z}+\nu_{,\varpi}\psi_{,\varpi}-\nu_{,z}\psi_{,z})-\tfrac{1}{2}e^{\beta4\nu}[\omega_{,\varpi}^{2}\omega_{2}^{2}] = 0. \tag{1.64}$$

These last three equations are of course redundant, because the Bianchi identities express linear combinations of them and their first derivatives in terms of equations (1.58)–(1.61).

(c) *Properties of stellar models*

For a one-parameter equation of state, $p = p(\epsilon)$, the uniformly rotating equilibria form a two-dimensional set, shown in figure 1 as a surface in ϵ_c–J–M space, with ϵ_c the central density, J the angular momentum and M the mass of a star. The part of

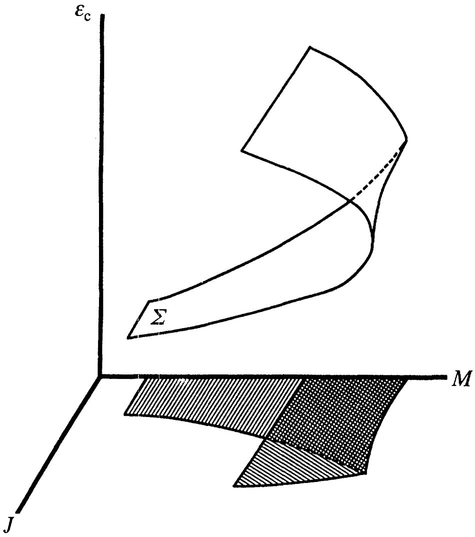

Figure 1. Equilibrium configurations form a two-dimensional set Σ, depicted here as a surface embedded in the space with coordinates J, M, and ϵ_c. The projection of Σ on the JM-plane doubles back on itself, leaving points to the right uncovered.

the surface where M increase monotonically with increasing ϵ_c is the region of stars stable against collapse (see §2*d*).

The boundary of this stable region has four parts. The angular momentum is bounded between $J = 0$ and a maximum value at fixed baryon number for which the equator of the star rotates at the Kepler frequency of a particle in circular orbit. The line of maximum central density is along the maximum-mass ridge. However, only part of the line of stable stars with minimum central density lies along a valley of minimum mass, where $\partial M(\epsilon_c, J)/\partial \epsilon_c = 0$. For rapidly rotating models, as one decreases ϵ_c at fixed J, the angular velocity of star's equator reaches the Kepler frequency before an extremal value of mass is attained. The corresponding moment of inertia is defined by

$$I = J/\Omega. \tag{1.65}$$

For a given equation of state, models with the largest values of angular velocity (and angular momentum) have the largest masses. These stars are supported by their rotation, and have baryon number larger than the upper limit for spherical stars. A star like this cannot lose its angular momentum without collapsing: its evolutionary path is a path of constant baryon number, and the star collapses when it reaches the maximum-mass ridge in the diagram.

Models of rotating relativistic stars have been computed by several different authors. Early codes were obtained by Bonazzola & Schneider (1974) and Wilson (1972). Butterworth & Ipser (1976), following a newtonian algorithm due to Stoeckly (1965), incorporated more precise asymptotic conditions, obtaining an accurate code that was used to construct polytropes and uniform-density configurations, the relativistic analogues of the newtonian Maclaurin spheroids. The code was modified by Friedman *et al.* (1986) to accommodate a set of proposed equations of state (EOS) for matter above nuclear density, and several hundred models were constructed to find the characteristics of rotating relativistic stars for a wide range of EOS. Additional models, including some based on an EOS for quark matter, were

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

constructed in the wake of the spurious observation of a 0.5 ms pulsar in SN1987 by these authors (Friedman *et al.* 1989) and by Lattimer *et al.* (1990), whose code was similarly based on the Butterworth–Ipser algorithm. Komatsu *et al.* (1989*a*) have recently obtained a successful code with a somewhat different algorithm. In the Butterworth–Ipser approach, the equations for each potential are discretized to give a matrix acting on the vector of values of that potential, and the matrix is inverted to solve the equation. Komatsu *et al.* separate off laplacian operators with constant coefficients which they invert by numerical integration of an analytic Green's function. They construct differentially rotating polytropes, including a set of toroidal configurations, and the method may be useful for modelling accretion disks about black holes. Finally, Cook *et al.* (1992) have constructed a set of polytropic models with a code based on the method of Komatsu *et al.*

Nearly all observed neutron stars are rotating slowly enough that they can accurately be modelled by using the Hartle's (1967) slow-rotation formalism. To $O(\Omega^2)$, the structure of the star can be found by integrating a set of ordinary differential equations. These are valid for $\Omega \ll \Omega_K$, the Kepler frequency of a satellite in circular orbit at the equator. A review of rotating neutron stars based on an extensive set of slowly rotating models can be found in Datta (1988). Recent work by Weber *et al.* (1991) and Weber & Glendenning (1991, 1992) uses Hartle's formalism to estimate properties of rapidly rotating stars. Although the radius and shape of a star rotating at nearly its maximum frequency depart substantially from the slow-rotation approximation, in numerical models based on the exact equations, the metric is near that given by the slow-rotation formalism. Weber *et al.* argue that the maximum frequency can be accurately approximated as well. Using (1.67) below, with metric and radius evaluated to $O(\Omega)$, they solve $\Omega = \Omega_K(\Omega)$.

The mass, baryon mass, and angular momentum of a rotating relativistic star are integrals over the fluid:

$$M = \int (T_{ab} - \tfrac{1}{2} g_{ab} T)\, t^a n^b \,\mathrm{d}V, \quad M_0 = \int \rho u_a n^a \,\mathrm{d}V, \quad J = \int T_{ab}\, \phi^a n^b \,\mathrm{d}V. \tag{1.66}$$

The maximum angular momentum for a uniformly rotating stellar model corresponds to an angular velocity equal to the Kepler frequency, Ω_K, the angular velocity of a particle in circular orbit at the equator. This has the form,

$$\Omega_K = \omega + \omega'/2\psi' + [\mathrm{e}^{2\nu - 2\psi}\nu'/\psi' + (\omega'/2\psi')^2]^{\frac{1}{2}}. \tag{1.67}$$

As a spherical star spins up, it becomes oblate, and the Kepler frequency of a particle at larger equatorial radius is correspondingly smaller. By the time the star itself rotates at the Kepler frequency, Ω_K has typically fallen to about 60% of its value for the spherical configuration with the same baryon number.

The uncertainty in the nuclear EOS leads to sharp differences in possible models of rotating stars. For the softest EOS, $1.4 M_{\odot}$ non-rotating models have radii of about 8 km, while models based on the stiffest EOS (consistent with the fastest observed pulsars) are much less centrally condensed, with non-rotating radii of about 15 km. For the corresponding rotating models, the ratio of the radii is not substantially different, about 12 km for the softest models compared with 20 km for the stiffest. The ratio of the moment of inertia, however, is enhanced by rotation, and this leads to the sensitivity, mentioned in the next section, of the maximum frequency of rotation on the EOS. Figure 2 displays moments of inertia for a set of equations of

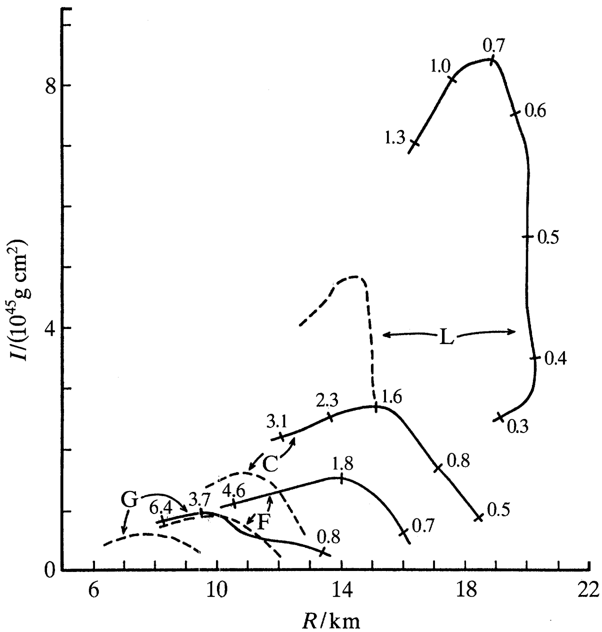

Figure 2. Moment of inertia is plotted against equatorial radius for sequences of models based on equations of state C, F, G, and L. for each EOS, a sequence of spherical models is represented by a dashed line, while a solid line represents models rotating at maximum (keplerian) angular velocity, $\Omega = \Omega_K$. Along each curve with $\Omega = \Omega_K$, tickmarks are labelled with the value of the model's central density in units of 10^{15} g cm^{-3}.

state that span the range of compressibilities from EOS G, slightly softer than needed to allow a $1.4M_\odot$ spherical star, to EOS L, slightly too stiff to allow a 642 Hz pulsar with a mass of $1.4M_\odot$. (The first constraint is clearly needed from the well-determined masses of binary pulsars. The constraint set by rotation assumes that the masses of the fastest pulsars are not much larger than $1.4M_\odot$.) Note that for each EOS, the configuration with maximum moment of inertia has less mass than the maximum allowed by the EOS.

The model with largest mass, however, is also the model with the largest baryon number, angular momentum, red- and blueshifts, and the largest value of the frame dragging frequency, ω among all uniformly rotating configurations stable against collapse. However, Shapiro *et al.* (1990) found for newtonian polytropes, that just before to the termination point of sequence of rotating models, where $\Omega = \Omega_K$, the angular velocity can turn over as a function of angular momentum. Cook *et al.* (1992) have recently extended the investigation to relativistic polytropes, for which the effect is more pronounced, but for compressibilities characteristic of neutron star matter the difference between Ω_K and the maximum angular velocity appears to be either small or non-existent. To within the accuracy of the numerical studies of 'realistic' equations of state proposed for neutron star matter (Friedman *et al.* 1986, 1989; Lattimer *et al.* 1990), there was no observed distinction between the model with maximum mass (and angular momentum) and a model with maximum angular velocity. No such distinction was sought, however, and it remains to be seen whether, for proposed neutron star equations of state, the configurations with maximum mass and angular velocity coincide.

(d) Limits on masses and rotation rates of relativistic stars

The precise upper mass limit on a relativistic star depends on the density below which one believes that the EOS is known. If one demands a one-parameter EOS, satisfying

$$dp/d\epsilon \geqslant 0, \tag{1.68}$$

and

$$dp/d\epsilon \leqslant 1, \tag{1.69}$$

then a spherical star with a standard EOS below ϵ_0 obeys the empirical formula (Hartle 1978),

$$M < 6.8(\epsilon_0/(10^{14}\ \mathrm{g\ cm^{-3}}))^{-\frac{1}{2}} M_\odot. \tag{1.70}$$

The apparent upper limit of about $3.2M_\odot$ on stellar models based on proposed equations of state for matter above nuclear density would correspond here to assuming that one knows the EOS below 4.5×10^{14} g cm^{-3} (Rhoades & Ruffini 1974). For uniformly rotating stars, the limit (1.70) is higher by 24–25%, over the maximum mass for non-rotating configurations (Friedman & Ipser 1987),

$$M < 8.4(\epsilon_0/(10^{14}\ \mathrm{g\ cm^{-3}}))^{-\frac{1}{2}} M_\odot. \tag{1.71}$$

Using two different candidates (Negele–Vauthrin & Baym–Pethick–Sutherland) for the 'known' EOS below nuclear density gave agreement to within 1% in the upper mass.

The causality constraint, (1.69), is not above suspicion. The quantity $(dp/d\epsilon)^{\frac{1}{2}}$ is the phase velocity of hydrodynamic waves in a neutron star, but it is not clear that microcausality or an analogous requirement on the group velocity enforces (1.69). The limit (1.71) is empirical, found to occur, as expected for the stiffest EOS consistent with (1.69), namely $p = \epsilon + \mathrm{const.}$, for $\epsilon \geqslant \epsilon_0$. There is as yet no proof that maximizing stiffness for rotating stars maximizes the upper mass. The change in mass for models based on EOS for nuclear matter is discussed below.

We emphasized in the last section the large differences in radius and moment of inertia of rapidly rotating models that have the same baryon number but are based on different EOS. The difference in the limiting angular velocity is equally dramatic. The most compact models (corresponding roughly to the softest equations of state) can rotate with angular velocities three times as large as models with the largest radii and stiffest EOS. If neutron stars spun up by accretion or formed by accretion-induced collapse of white dwarfs can rotate at their limiting angular velocity, then one can hope to observe that limit; to see an excess of pulsars rotating with nearly the velocity of the fastest observed pulsar (Friedman 1973; Friedman *et al.* 1986; Imamura *et al.* 1987). Knowing the limiting frequency would sharply constrain the EOS above nuclear density.

At present the two fastest pulsars have frequencies within 3% of each other, at 4033 and 3910 s^{-1}. Because the magnetic fields of the two pulsars differ by a factor of at least 1.8, if the pulsar's frequency is limited by their magnetic field rather than by gravity, the 3% agreement would have to be coincidental. If they are in fact rotating at or near their limiting frequencies, and if their masses are at least $1.4M_\odot$, then the EOS above nuclear density must be unexpectedly stiff (Lipunov & Postnov 1988; Friedman *et al.* 1988).

As is discussed in §2*c*, the maximum frequency can depend on whether a pulsar is spun up by accretion or is formed from accretion induced collapse. In the former case the Kepler frequency is likely to set the limit; but the angular velocity of a

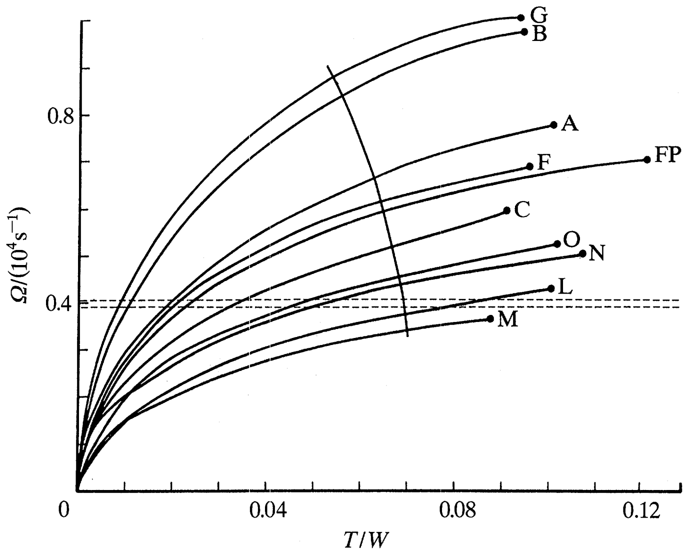

Figure 3. Sequences of uniformly rotating neutron stars of baryon mass $M_B = 1.4M_\odot$ for equations of state tabulated by Arnett & Bowers (indicated by letters), together with the more recent Friedman–Pandharipande equation of state (FP). The keplerian termination point of each sequence is indicated by a dot. The diagonal line crossing the sequences is a conservative estimate (that is, an underestimate) of the smallest rotation for which the models could be unstable to non-axisymmetric perturbations. The Ω values for PSR1937+214 and PSR1957+20 are indicated by the upper and lower dashed lines, respectively.

neutron star formed in the collapse of a white dwarf can be limited by a non-axisymmetric instability. It appears, however, that the two limiting frequencies are close to one another. For newtonian models, Lindblom & Ipser find that the non-axisymmetric stability limit, Ω_{lim} is within 10% of Ω_{K}.

We expect a similar result in the relativistic case, and the reason is suggested by figure 3. The figure is a slight modification by Imamura *et al.* (1988) of a graph from Friedman *et al.* (1986). Each line corresponds to the sequence of neutron stars with baryon mass approximately $1.4M_\odot$ and EOS labelled by a letter next to the termination point of the sequence (at $\Omega = \Omega_{\text{K}}$). Angular velocity is plotted against the parameter $t = T/|W|$, the ratio of rotational energy to potential energy of rotating stars. The diagonal curve crossing the sequences underestimates (we believe) the value of t for which the models could be unstable to non-axisymmetric perturbations. The estimate is based on the dependence of t on the polytropic index for newtonian stars. The key feature of the curves is their flatness near the termination points. Even if the instability point had a value of $T/|W|$ as small as that shown, Ω_{lim} would not differ by more than about 15% from Ω_{K}. In addition, the stability (under)estimate here and the computation of Lindblom & Ipser in the newtonian case both used a viscosity that ignored the bulk viscosity recently discussed by Sawyer and an effective viscosity arising from the interaction of electrons with superfluid vortices discussed by Lindblom & Mendell (see §2*d*).

From figure 3 it is apparent that only the stiffer EOS are consistent with a limiting frequency near 4033 s^{-1}. If one assumes that the frequency of each pulsar is within 25% of the limiting value, then of 11 EOS used in the tabulated models of Arnett & Bowers, only four models (L, M, N and O) are consistent with $M > 1.4M_\odot$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Table 1. *Models with maximum mass and rotation for the equations of state mentioned in the text*

equation of state	Ω	ϵ_c	$M/M_\odot$	(%)	$M_0/M_\odot$	R	T/W	I	cJ/GM^2	e	β
L	0.76	1.11	3.18	(20)	3.72	17.3	0.122	7.87	0.68	0.69	0.34
PAL1	1.00	3.11	1.73	(15)	1.93	13.0	0.090	1.54	0.58	0.70	0.45
D	1.04	2.78	1.94	(17)	2.21	12.7	0.11	2.00	0.63	0.69	0.40
C	1.11	2.71	2.16	(17)	2.47	12.9	0.11	2.42	0.64	0.68	0.35
PAL3	1.16	3.98	1.65	(15)	1.85	11.3	0.094	1.23	0.59	0.69	0.47
FP	1.23	2.5	2.30	(17)	2.71	12	0.13	2.41	0.70	0.67	0.28
F	1.24	4.1	1.66	(13)	1.87	11	0.094	1.16	0.60	0.67	0.39
A	1.28	3.29	1.94	(17)	2.25	10.8	0.117	1.71	0.66	0.67	0.33
π	1.54	4.47	1.74	(15)	2.02	9.18	0.121	1.15	0.66	0.66	0.30
B	1.57	5.16	1.65	(17)	1.91	9.2	0.107	0.98	0.64	0.66	0.31
G	1.52	5.5	1.55	(14)	1.73	8.6	0.101	0.86	0.62	0.62	0.34

The properties listed are Ω, angular velocity in $10^4\ \mathrm{s}^{-1}$, ϵ_c, central density in $10^{15}\ \mathrm{g\ cm^{-3}}$, $M/M_\odot$ and (%), gravitational mass and its percentage increase over the maximum mass of the spherical model; $M_0/M_\odot$, baryon mass; R, equatorial radius [(proper circumference)/2π] in km; T/W, ratio of rotational energy to gravitational energy; I, moment of inertia in $10^{45}\ \mathrm{g\ cm^2}$, cJ/GM^2, dimensionless ratio of angular momentum J to M^2, e, eccentricity; and β, injection energy.

Extensive calculations of the equilibria of rapidly rotating relativistic stellar models (Friedman *et al.* 1986, 1989; Lattimer *et al.* 1990) have yielded considerable insight into the structural properties of neutron-star models, including their dependence on the assumed form of the high-density equation of state. In the present brief discussion of these properties, we shall use as a framework the discussion of Friedman *et al.* (1989), with particular focus on the information contained in table 1 and figure 1 of that reference, which are reproduced here as table 1 and figure 4. The data restrict attention to uniform rotation, as is appropriate in the presence of viscosity, and contain information about a particular complex of representative EOS that have been proposed for neutron stars. Most of these EOS are from the Arnett & Bowers (1977) collection and are identified by the symbols used in that reference. They span the range from very soft to very stiff. Added to these are EOS proposed by Friedman & Pandharipande (1981; EOS denoted by FP), by Prakash *et al.* (1988; PAL1 for their function $F(u) = u$; PAL3 for $F(u) = \sqrt{u}$), and by Weise & Brown (1975; π).

For each EOS, the key uniformly rotating model is the one with maximum gravitational mass and baryon mass. This model rotates with essentially the maximum angular velocity $\Omega_{\max}$, for the given EOS, that is consistent with uniform rotation and stability against gravitational collapse. Properties of these maximum-mass models are exhibited in table 1. One evident feature is that softer EOS allow larger maximum angular velocities. This is because the dynamical timescale, as determined by the inverse square-root of the average density, is shorter in the corresponding configurations. In fact, close examination reveals that

$$\left(\frac{\Omega_{\max}}{10^4\ \mathrm{s}^{-1}}\right) \approx 0.76 \left(\frac{M_s}{M_\odot}\right)^{\frac{1}{2}} \left(\frac{R_s}{10\ \mathrm{km}}\right)^{-\frac{3}{2}} \tag{1.72}$$

to better than 7% accuracy (Haensel & Zdunik 1989; Friedman *et al.* 1989). Here M_s and R_s are the mass and radius of the maximum-mass non-rotating configuration for the given equation of state. Rotation increases the maximum mass above M_s by

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

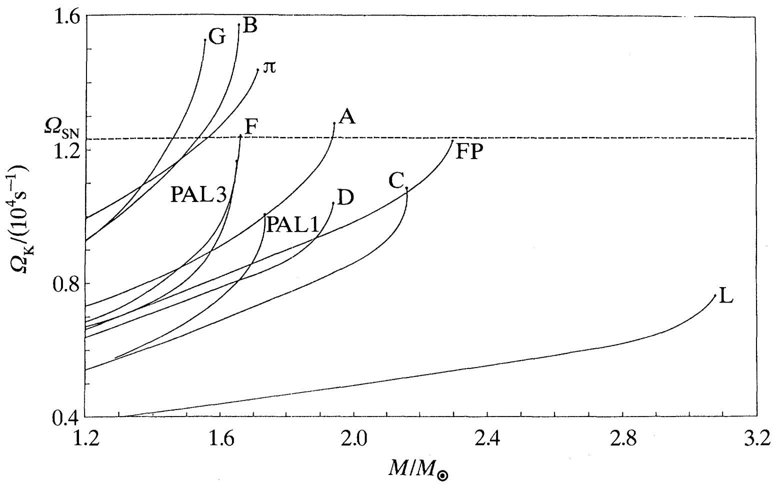

Figure 4. Maximum angular velocity Ω_K against M for 11 EOS. Single letters labelling curves follow the notation of Arnett & Bowers. FP refers to the Friedman–Pandharipande equation of state, π to an EOS with pion condensation due to Weise & Brown, and PAL1 and PAL3 are equations of state in the parametrized family considered by Prakash, Ainsworth and Lattimer (see text). For each equation of state, the final dot represents the configuration with maximum mass and with the maximum angular velocity consistent with stability against collapse. The dotted line marks the frequency, $\Omega_{SN} = 1.237 \times 10^4\ s^{-1}$, observed in SN 1987A.

ca. 15–20 %. This has the interesting consequence that rotating models with $M > M_s$ exist and are stable if their rotation rates are sufficiently high. If such a model spins down, say by radiating away angular momentum via gravitational waves, it must eventually reach a point at which it becomes unstable and undergoes gravitational collapse.

Another evident fact is that few EOS simultaneously satisfy $\Omega_{max} \gtrsim 1.25 \times 10^4\ s^{-1}$ and $M_s \geqslant 1.44 M_\odot$. This latter condition is required to accommodate the measured mass of the binary pulsar PSR 1913+16 (Taylor & Weisberg 1989), and it is satisfied only by EOS that are sufficiently hard. Because of this, and because of the fact that Ω_{max} decreases as hardness increases, observation of a value of $\Omega \gtrsim 1.25 \times 10^4\ s^{-1}$ would rule out almost all EOS that have been proposed. A final point about table 1 is that most EOS yield ratios of rotational energy to potential energy that satisfy $T/W \lesssim 0.12$. Hence it appears unlikely that non-axisymmetric Jacobi-like configurations in uniform rotation can form. This is unfortunate from the point of view that such configurations could be strong emitters of gravitational waves (cf. Ipser & Managan 1981). As has been pointed out elsewhere (Glendenning 1990), an exception to the rather low limit on T/W occurs for the case of stars envisaged as made up of strange-quark matter. In such objects the pressure vanishes at a density of the order of nuclear densities. This yields configurations with small radii for all stellar masses available and permits large values $T/W \approx 0.18$ and $\Omega_{max} \gtrsim 1.6 \times 10^4\ s^{-1}$, depending on the density at which pressure vanishes.

Figure 4 exhibits a plot of the maximum angular velocity Ω_K that can be exhibited by models of a given mass, for each EOS. The curve for each EOS terminates at the maximum-mass model of table 1. For reference, the dotted line marks the spurious angular velocity Ω_{SN} once attributed to SN 1987A, the recent Magellanic Cloud supernova.

The stiffer EOS have $\Omega_{\max} < \Omega_{\mathrm{SN}}$. In fact the only EOS in the figure that simultaneously yield $\Omega_{\max} > \Omega_{\mathrm{SN}}$ and $M_{\mathrm{s}} > 1.44 M_{\odot}$ are π, A, F and FP, with the latter two barely making it. Notice that for a fixed mass M, Ω_{K} decreases as the stiffness of the EOS increases. Hence, if the mass M of a millisecond pulsar could be pinned down, its observed rotation rate could be used to rule out all of the stiffer EOS for which $\Omega_{\mathrm{K}}(M)$ is less than the observed rate. Also, any observed value of $\Omega \gtrsim 1.6 \times 10^4\ \mathrm{s}^{-1}$ would be hard to explain other than in terms of strange-quark models or something similar.

2. Stellar oscillations and stability

(a) *Perturbation theory of relativistic fluids*

(i) *Lagrangian and eulerian perturbations*

Perturbations of rotating stars have been discussed in the context of general relativity by a number of authors. The present discussion of perfect fluids is essentially the formalism of Friedman & Schutz (1975) (see also Friedman 1978), and it is closely related to work by Taub (1969), Schutz (1972), Chandrasekhar & Friedman (1973*a*, *b*), Carter (1973) and Schutz & Sorkin (1977). The treatment of imperfect fluids is based on that given by Lindblom & Hiscock (1983).

In discussing stellar oscillations one is interested in the time evolution of nearby configurations having the same baryon number and the same total entropy, configurations that can be viewed as deformations of the original equilibrium. Formally, we introduce a family of (time-dependent) solutions

$$Q(\lambda) = \{g_{ab}(\lambda), u^a(\lambda), \epsilon(\lambda), p(\lambda)\} \tag{2.1}$$

indexed by a parameter λ, and compare, to first order in λ, the perturbed variables $Q(\lambda)$ with their equilibrium values, $Q(0)$. We further suppose that the family of solutions $Q(\lambda)$ is such that each member can be reached by an adiabatic deformation of the equilibrium $Q(0)$. That is, there is to be a family of diffeomorphisms χ_λ mapping fluid trajectories of the equilibrium model $Q(0)$ to fluid trajectories of the solution $Q(\lambda)$.

First-order departures from equilibrium can be described in two ways. The eulerian perturbations in the quantities $Q(\lambda)$ are defined by

$$\delta Q = \frac{\mathrm{d}}{\mathrm{d}\lambda} Q(\lambda)\,|_{\lambda=0} \tag{2.2}$$

and compare values of Q at the same point of the space-time. In the region occupied by the original fluid, one can also introduce the lagrangian perturbations

$$\Delta Q = \frac{\mathrm{d}}{\mathrm{d}\lambda}[\chi_{-\lambda} Q(\lambda)]\,|_{\lambda=0} = (\delta + \mathcal{L}_\xi)\, Q, \tag{2.3}$$

where ξ^a generates the family of diffeomorphisms χ_λ (that is, the curve $\lambda \to \chi_\lambda(P)$ has tangent $\xi^a(P)$ at the point P). The field ξ^a is termed a lagrangian displacement and may be regarded as the connecting vector joining fluid elements in the unperturbed configuration to the corresponding elements in the perturbed space-time.

The first order changes in the variables Q can be expressed in terms of the displacement ξ^a and the eulerian change in the metric

$$h_{ab} = \delta g_{ab}. \tag{2.4}$$

We use the relations
$$\Delta g_{ab} = h_{ab} + \nabla_a \xi_b + \nabla_b \xi_a, \tag{2.5}$$
$$\Delta \epsilon_{abcd} = \tfrac{1}{2}\epsilon_{abcd}\, g^{ef}\, \Delta g_{ef}. \tag{2.6}$$
Requiring that world lines of the unperturbed configuration are mapped by χ_λ to world lines of the perturbed fluid implies
$$\Delta u^a = \tfrac{1}{2} u^a u^b u^c \,\Delta g_{bc}, \tag{2.7a}$$
or
$$\delta u^a = q^a{}_b \,\mathscr{L}_u\, \xi^b + \tfrac{1}{2} u^a u^b u^c h_{bc}. \tag{2.7b}$$

The perturbed energy conservation equation, (1.13), takes the form
$$0 = \Delta(u_a \nabla_b T^{ab}) = -\Delta[(\epsilon+p)\nabla_b u^b + u^b \nabla_b \epsilon] = -u^b \nabla_b[\Delta\epsilon + \tfrac{1}{2}(\epsilon+p)\, q^{ab}\,\Delta g_{ab}] \tag{2.8}$$
with first integral
$$\Delta\epsilon = -\tfrac{1}{2}(\epsilon+p)\, q^{ab}\,\Delta g_{ab}. \tag{2.9}$$
This expresses the fact that the flow is isentropic. In terms of the comoving baryon density, n, the second law of thermodynamics, (1.18), implies
$$(\Delta\epsilon)/(\epsilon+p) = (\Delta n)/n. \tag{2.10}$$
Conservation of baryons implies that the fractional increase in baryon density is the fractional decrease in comoving volume,
$$(\Delta n)/n = -\tfrac{1}{2} q^{ab}\,\Delta g_{ab} \tag{2.11}$$
and (2.9) follows from (2.10) and (2.11). The lagrangian change in the pressure is similarly given by
$$\Delta p = \gamma p(\Delta\epsilon)/(\epsilon+p) = -\tfrac{1}{2}\gamma p q^{ab}\,\Delta g_{ab}, \tag{2.12}$$
where the adiabatic index γ is defined by
$$\gamma = \frac{\partial \ln p(n,s)}{\partial \ln n} = \frac{\epsilon+p}{p}\frac{\partial}{\partial\epsilon} p(\epsilon, s). \tag{2.13}$$

(ii) *An action for the perturbation equations*

The equations governing perturbations of a perfect fluid,
$$\delta(G^{ab} - 8\pi T^{ab}) = 0, \tag{2.14}$$
$$\delta \nabla_b T^{ab} = 0 \tag{2.15}$$
are a self-adjoint system. For any pairs (ξ^a, h_{ab}) and $(\hat{\xi}^a, \hat{h}_{ab})$, self-adjointness (in the weak sense of a symmetric system) has the form
$$\hat{\xi}_b\,\delta(\nabla_c T^{bc}) + (1/16\pi)\,\hat{h}_{bc}\,\delta(G^{bc} - 8\pi T^{bc}) = -\mathscr{L}(\hat{\xi}, \hat{h}; \xi, h) + \nabla_b R^b, \tag{2.16}$$
where $\mathscr{L}$ is symmetric under interchange of (ξ, h) and $(\hat{\xi}, \hat{h})$, and where $(\Delta u^a, \Delta\epsilon, \Delta p)$ and $(\hat{\Delta} u^a, \hat{\Delta}\epsilon, \hat{\Delta} p)$ are given in terms of (ξ, h), $(\hat{\xi}, \hat{h})$, respectively, by (2.7), (2.9) and (2.12). Because (2.8) is satisfied, $u_a\,\delta(\nabla_b T^{ab}) = 0$, and the component of ξ^a along u^a does not appear on the left-hand side of (2.16).

Explicitly,
$$\begin{aligned}\mathscr{L}(\hat{\xi}, \hat{h}; \xi, h) = {} & U^{abcd}\,\nabla_a \hat{\xi}_b \nabla_c \xi_d + V^{abcd}(\hat{h}_{ab}\nabla_c \xi_d + h_{ab}\nabla_c \hat{\xi}_d) \\ & -(1/32\pi)\,\epsilon^{aceg}\epsilon^{bdf}{}_g\,\nabla_c \hat{h}_{ab}\nabla_d h_{ef} \\ & - T^{ab} R_{acbd}\,\hat{\xi}^c \xi^d + (\tfrac{1}{2}W^{abcd} - (1/16\pi)\,G^{abcd})\,\hat{h}_{ab} h_{cd} \\ & -\tfrac{1}{2}\nabla_c T^{ab}(\hat{h}_{ab}\xi^c + h_{ab}\hat{\xi}^c),\end{aligned} \tag{2.17}$$

and
$$R^a(\hat{\xi},\hat{h};\xi,h) = \hat{\xi}_b \Pi^{ab} + \hat{h}_{bc}\pi^{abc}, \tag{2.18}$$
where
$$\Pi^{ab} = \frac{1}{2}\frac{\partial \mathscr{L}(\xi,h;\xi,h)}{\partial \nabla_a \xi_b} = U^{abcd}\nabla_c \xi_d + V^{cdab}h_{cd}, \tag{2.19}$$
$$\pi^{abc} = \frac{1}{2}\frac{\partial \mathscr{L}(\xi,h;\xi,h)}{\partial \nabla_a h_{bc}} = -\frac{1}{32\pi}\epsilon^{aeg(b}\epsilon^{c)df}{}_g \nabla_b h_{ef}, \tag{2.20}$$
$$G^{abcd} = \tfrac{1}{2}R^{a(cd)\,b} + \tfrac{1}{4}(2R^{ab}g^{cd} + 2R^{cd}g^{ab} - 3R^{a(c}g^{d)\,b} - 3R^{b(c}g^{d)\,a}) + \tfrac{1}{4}R(g^{ac}g^{bd} + g^{ad}g^{bc} - g^{ab}g^{cd}), \tag{2.21}$$
$$U^{abcd} = (\epsilon+p)\,u^a u^c q^{bd} + p(g^{ab}g^{cd} - g^{ad}g^{bc}) - \gamma p q^{ab} q^{cd}, \tag{2.22}$$
and
$$2V^{abcd} = (\epsilon+p)\,(u^a u^c q^{bd} + u^b u^c q^{ad} - u^a u^b q^{cd}) - \gamma p q^{ab} q^{cd}. \tag{2.23}$$

The derivation is given in detail in Friedman & Schutz (1975), but a generalization will be sketched below. It extends to relativity the eulerian variational principle obtained in the newtonian framework by Ipser & Managan and in the relativistic Cowling approximation by Ipser & Lindblom. First note that (2.16) implies that an action for the perturbation equations (2.14) and (2.15) is given by
$$I = \int \mathrm{d}\tau \tfrac{1}{2}\mathscr{L}(\xi,h;\xi,h). \tag{2.24}$$

That is, by introducing potentials ξ^a for the perturbations we obtain an unconstrained action. The price is an additional gauge freedom (Soper 1976; Carter & Quintana 1976; Schutz & Sorkin 1977), a set of trivial displacements η^a for which the physical perturbation, δu^a, $\delta\epsilon$, δp, h_{ab} vanishes. Trivial displacements η^a are of the form
$$\eta^a = n^{-1}\epsilon^{abc}\nabla_b h \nabla_c f + g u^a \tag{2.25}$$
with f and h any scalars for which $u^a\nabla_a h = u^a \nabla_a f = 0$ and for which $\nabla_a h$ is along $\nabla_a s$ when $\nabla_a s \neq 0$.

For perturbations with harmonic time dependence (e.g. outgoing modes),
$$\xi^a = \tilde{\xi}^a\,\mathrm{e}^{\mathrm{i}(m\phi+\sigma t)}, \quad h_{ab} = \tilde{h}_{ab}\,\mathrm{e}^{\mathrm{i}(m\phi+\sigma t)}, \tag{2.26}$$
with $\tilde{\xi}$ and $\tilde{h}$ independent of t and ϕ, we have (see (1.53))
$$q^a{}_b \mathscr{L}_u \xi^b = \mathrm{i}u^t(\sigma + m\Omega)\,\xi^a$$
and, as long as $\sigma + m\Omega \neq 0$ (the case of a trivial displacement), (2.7b) can be inverted to give
$$\xi^a = [\mathrm{i}u^t(\sigma+m\Omega)]^{-1}(\delta u^a - \tfrac{1}{2}u^a u^b u^c h_{bc}). \tag{2.27}$$
With ξ^a replaced in this way by δu^a and h_{bc}, and with $\delta\epsilon$ and δp defined in terms of δu^a and h_{ab} by (2.9) and (2.12), the description of the perturbation is purely eulerian.

As Ipser and his co-workers have observed, however, one can organize the equations differently to exploit the fact that, for a mode, δu^a occurs algebraically in the perturbed Euler equations,
$$q^a{}_c\,\delta\nabla_b T^{bc} = 0 \tag{2.28}$$
(Ipser & Managan 1985; Ipser & Lindblom 1989, 1990, 1992). If one initially leaves the energy equation, (2.9) unsolved, so that
$$\zeta \coloneqq -\frac{1}{\mathrm{i}u^t(\sigma+m\Omega)}u_a\,\delta(\nabla_b T^{ab}) \tag{2.29}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

is not initially set to zero, then the Euler equations (2.28) can be solved algebraically for δu^a in terms of δp and h_{ab}.

A generalization of the symmetry relation (2.16) that includes $u_b\,\delta\nabla_a T^{ab}$ has the form

$$\hat{\xi}_b\,\delta(\nabla_c T^{bc}) + (1/16\pi)\,\hat{h}_{bc}\,\delta(G^{bc} - 8\pi T^{bc}) = -\tilde{\mathscr{L}}(\hat{\xi}, \hat{h}, \xi, h) + \nabla_b \tilde{R}^b, \tag{2.30}$$

where

$$\xi^a u_a = \frac{1}{\mathrm{i}u^t(\sigma + m\Omega)}\left(\frac{\delta p}{\epsilon + p} - \tfrac{1}{2}u^a u^b h_{ab}\right), \tag{2.31}$$

and

$$\tilde{\mathscr{L}}(\hat{\xi}, \hat{h}, \xi, h) = \mathscr{L}(\hat{\xi}_\perp, \hat{h}, \xi_\perp, h) - (\gamma p/(\epsilon + p)^2)\,\hat{\zeta}\zeta, \tag{2.32}$$

with $\xi^a_\perp = q^a{}_b\,\xi^b$ the projection of ξ^a orthogonal to u^a and ζ given by (2.29). Equation (2.30) is obtained as follows. The perturbed Einstein tensor has the form

$$(1/\sqrt{-g})\,\delta(\sqrt{-g}\,G^{ab}) = -\tfrac{1}{2}\epsilon^{aceg}\epsilon^{bdf}{}_g\nabla_{(c}\nabla_{d)}\,h_{ef} + G^{abcd}h_{cd}, \tag{2.33}$$

with the symmetry

$$\hat{h}_{ab}(1/\sqrt{-g})\,\delta(\sqrt{-g}\,G^{ab}) = \tfrac{1}{2}\epsilon^{aceg}\epsilon^{bdf}{}_g\nabla_a\,\hat{h}_{cd}\nabla_b\,h_{ef} + G^{abcd}\hat{h}_{ab}\,h_{cd} - \nabla_a(\tfrac{1}{2}\epsilon^{aceg}\epsilon^{bdf}{}_g\,\hat{h}_{cd}\nabla_b\,h_{ef}). \tag{2.34}$$

If (2.9) and (2.12) were used to express $\Delta\epsilon$ and Δp in terms of ξ^a, we would have

$$(1/\sqrt{-g})\,\Delta(\sqrt{-g}\,T^{ab}) = W^{abcd}\Delta g_{cd}, \tag{2.35}$$

with

$$W^{abcd} = \tfrac{1}{2}(\epsilon + p)\,u^a u^b u^c u^d + \tfrac{1}{2}p(g^{ab}g^{cd} - g^{ac}g^{bd} - g^{ad}g^{bc}) - \tfrac{1}{2}\gamma p q^{ab}q^{cd} = W^{cdab}, \tag{2.36}$$

and corresponding symmetry

$$\hat{\Delta}g_{ab}\frac{1}{\sqrt{-g}}\Delta(\sqrt{-g}\,T^{ab}) = W^{abcd}\,\hat{\Delta}g_{ab}\,\Delta g_{cd}. \tag{2.37}$$

When (2.9) is not used, ζ of (2.29) is non-zero and one must write

$$\Delta\epsilon = \zeta - \tfrac{1}{2}(\epsilon + p)\,q^{ab}\,\Delta g_{ab},$$
$$\Delta p = (\gamma p/(\epsilon + p))\,\Delta\epsilon,$$

resulting in an additional term,

$$\left(u^a u^b + \frac{\gamma p}{\epsilon + p}q^{ab}\right)\hat{\Delta}g_{ab}\,\zeta = -2\left(\frac{\hat{\delta}p}{\epsilon + p} - \tfrac{1}{2}u^a u^b \hat{h}_{ab}\right)\zeta + 2\,\frac{\gamma p}{(\epsilon + p)^2}\hat{\zeta}\zeta,$$

added to the right-hand side of (2.37). Then, by using (2.31), (2.37) is replaced by

$$\hat{\Delta}g_{ab}(1/\sqrt{-g})\,\Delta(\sqrt{-g}\,T^{ab}) + 2\xi^c u_c\,u_a\nabla_b\,\delta T^{ab} = W^{abcd}\,\hat{\Delta}g_{ab}\,\Delta g_{cd} + 2(\gamma p/(\epsilon + p)^2)\,\hat{\zeta}\zeta. \tag{2.38}$$

Finally adding $1/16\pi$ (equation (2.34)) to (2.38) yields, after some algebra, the symmetry relation (2.30). The functionals $\mathscr{L}(\hat{\xi}, \hat{h}; \xi, h)$ and $\mathscr{L}(\hat{\xi}_\perp, \hat{h}; \xi_\perp, h)$ differ by a divergence which has been absorbed in $\nabla_a \hat{R}^a$.

(iii) *Energy and angular momentum*

Dynamical stability of a rotating star is governed by the sign of the energy of its perturbations. The canonical energy and angular momentum can be written as functionals quadratic in the perturbation. On a spacelike hypersurface S, with unit normal n^a, the canonical momenta conjugate to ξ^a and h_{ab} are given in terms of the quantities Π^{ab} and π^{abc} of (3.19) and (3.20) by,

$$\Pi^a = n_c\,\Pi^{ca}, \quad \pi^{ab} = n_c\,\pi^{cab}. \tag{2.39}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Corresponding to the Killing vector t^a is the conserved current,

$$j_t^c = \Pi^{ca} \mathscr{L}_t \xi_a + \pi^{cab} \mathscr{L}_t h_{ab} - \tfrac{1}{2} t^a \mathscr{L}, \tag{2.40}$$

and the corresponding canonical energy has the form (Friedman 1978),

$$E = \int_S \mathrm{d}S_a j_t^a = \int_S \mathrm{d}V (\Pi^a \mathscr{L}_t \xi_a + \pi^{ab} \mathscr{L}_t h_{ab} - \tfrac{1}{2} n t^a \mathscr{L}). \tag{2.41}$$

Explicitly,

$$E = \int_S \mathrm{d}S_e \{ U^{ebcd} \dot{\xi}_b \nabla_c \xi_d + V^{cdeb} h_{cd} \dot{\xi}_b - (1/32\pi) \epsilon^{ecfh} \epsilon^{bdg}{}_h \dot{h}_{cd} \nabla_b h_{fg} - \tfrac{1}{2} n t^e [U^{abcd} \nabla_a \xi_b \nabla_c \xi_d + 2V^{cdab} h_{cd} \nabla_a \xi_b - (1/32\pi) \epsilon^{acfh} \epsilon^{bdg} \nabla_a h_{cd} \nabla_b h_{fg} - T^{ab} R_{acbd} \xi^c \xi^d + \tfrac{1}{2} (W^{abcd} - (1/16\pi) G^{abcd}) h_{ab} h_{cd} - \nabla_c T^{ab} h_{ab} \xi^c] \}. \tag{2.42}$$

The canonical angular momentum corresponding to the Killing vector ϕ^a is similarly

$$J = \int_S \mathrm{d}S_a j_\phi^a = - \int_S \mathrm{d}V (\Pi^a \mathscr{L}_\phi \xi_a + \pi^{ab} \mathscr{L}_\phi h_{ab}). \tag{2.43}$$

The fact that the currents j_t and j_ϕ are conserved follow from the symmetry relation.

Along a family S_t of asymptotically spacelike hypersurfaces, energy and momentum are conserved. Along a family S_t of asymptotically null hypersurfaces, however, they change in time due to the radiation of energy and angular momentum at future null infinity. This radiated energy and angular momentum may again be expressed as surface integrals of j_t^a or j_ϕ^a, this time at null infinity. In particular, the canonical energy decreases monotonically from one asymptotically null hypersurface S_t to another S_t' in its future.

Although the currents are gauge dependent, the integrals E and J are gauge-invariant for gauge transformations,

$$\delta Q \rightarrow \delta Q + \mathscr{L}_\zeta Q,$$

that leave the perturbed metric asymptotically regular. The integrals are not invariant, however, under the additional gauge freedom associated with the trivial displacements of (3.23). This is related to the fact that E and J agree with the second-order change in energy and angular momentum of the equilibrium star only if the circulation of each ring of fluid is unchanged by the perturbation (Friedman & Schutz 1978*a*). One defines a class of canonical displacements by the requirement

$$q_a{}^c q_b{}^d \Delta \omega_{cd} = 0, \tag{2.44}$$

in order to compute E and J.

(*b*) *Oscillations of rotating stars*

(i) *Introduction*

Because of its importance for astrophysics in a variety of contexts, the problem of the oscillations of rotating newtonian stellar models has received considerable attention. Even so, up until very recently all attempts to obtain solutions to the equations governing the oscillations of rotating models had been unsuccessful. The lone exception is Clement's (1981) analysis of certain axisymmetric normal modes.

In large part, the reason why the newtonian oscillation problem has posed such difficulty is that the eigenequations for normal-mode pulsations are traditionally written down in terms of the lagrangian displacement vector ξ^a. This leads to a complicated eighth-order system of equations for four dependent variables. This system has proved intractable for generally non-axisymmetric modes. Although it has been used to develop variational principles (Lynden-Bell & Ostriker 1967; Friedman & Schutz 1978) for estimating normal-mode eigenfrequencies, it has yielded little information concerning the eigenfunctions themselves. And in a variety of contexts, detailed knowledge of the eigenfunctions is needed. For example, knowledge of the normal-mode eigenfunctions, in addition to the eigenfrequencies, is required for accurate assessment of the combined effects of gravitational radiation reaction and of viscous dissipation on the evolution and stability of millisecond pulsars, and for understanding the interaction between a pulsating star and a surrounding accretion disc.

We describe here a method developed recently (Ipser & Managan 1985; Managan 1985; Ipser & Lindblom 1990, 1991*a*, *b*) for solving the normal-mode equations of rapidly rotating newtonian stellar models. This method involves a reformulation of the stellar-pulsation equations in terms of two potential functions: $\delta\Phi$, the eulerian perturbation of the gravitational potential, and δU, the difference between the eulerian perturbation of the enthalpy and $\delta\Phi$. When the eigenequations are rewritten in terms of δU and $\delta\Phi$, a relatively simple fourth-order system of equations is obtained. This system has been solved successfully for both the eigenfrequencies and eigenfunctions of normal modes.

(ii) *The two-potential formalism*

The basic equations governing the evolution of a newtonian fluid configuration are the continuity equation, Euler's equation, and Poisson's equation:

$$\partial\rho/\partial t+\nabla_a(\rho v^a) = 0, \tag{2.45}$$

$$\rho(\partial v^a/\partial t+v^b\nabla_b v^a) = -\nabla^a p+\rho\nabla^a\Phi, \tag{2.46}$$

$$\nabla^a\nabla_a\Phi = -4\pi G\rho. \tag{2.47}$$

Here the variables ρ, v^a, p, and Φ are the mass density, velocity vector, pressure, and gravitational potential, respectively; G is Newton's constant; ∇_a is the standard euclidean covariant derivative. Throughout this subsection tensor indices are raised and lowered with the three-dimensional euclidean metric g_{ab} and its inverse g^{ab}. (In cartesian coordinates, g_{ab} is the identity matrix.)

We are interested in the pulsations of an equilibrium stellar model that is axisymmetric and rotating, perhaps differentially, about its z-axis. Hence the unperturbed velocity field is of the form $v^a = \Omega\phi^a$, where Ω is the equilibrium angular velocity and ϕ^a is a rotational Killing vector field that satisfies

$$\nabla_a\phi_b+\nabla_b\phi_a = 0. \tag{2.48}$$

Our demand that meridianal circulation be absent from the equilibrium state implies that (2.46) has the first integral

$$\tfrac{1}{2}\Omega^2\nabla_a\varpi^2 = (\nabla_a p)/\rho-\nabla_a\Phi, \tag{2.49}$$

where ϖ is the standard cylindrical radial coordinate.

Given an equilibrium configuration, we focus on small perturbations of its structure and motions away from equilibrium. We obtain the evolution equations governing these perturbations by linearizing equations (2.45)–(2.47) in the perturbations. This yields the perturbed equations

$$\partial\delta\rho/\partial t + v^a \nabla_a \delta\rho + \nabla_a(\rho\delta v^a) = 0, \tag{2.50}$$

$$\partial\delta v^a/\partial t + v^b \nabla_b \delta v^a + \delta v^b \nabla_b v^a = -(\nabla^a \delta p)/\rho + \delta\rho(\nabla^a p)/\rho^2 + \nabla^a \delta\Phi, \tag{2.51}$$

$$\nabla_a \nabla^a \delta\Phi = -4\pi G\,\delta\rho. \tag{2.52}$$

As usual, the symbol δ denotes the eulerian perturbation of a quantity. Quantities not preceded by a δ are understood to be equilibrium values. We complete the system of perturbed equations by assuming that the lagrangian change in pressure is proportional to the lagrangian change in the density:

$$\Delta p = \delta p + \xi^a \nabla_a p = (\gamma p/\rho)\,\Delta\rho = (\gamma p/\rho)(\delta\rho + \xi^a \nabla_a \rho), \tag{2.53}$$

where γ is the adiabatic index, and ξ^a is the lagrangian displacement. The vector ξ^a is related to the velocity perturbation by the relation

$$\delta v^a = \partial\xi^a/\partial t + v^b \nabla_b \xi^a - \xi^b \nabla_b v^a. \tag{2.54}$$

The traditional analysis of the perturbation equations involves eliminating δv^a in favour of ξ^a. The alternative method that has been developed recently proceeds in a different way, by eliminating ξ^a in favour of δv^a, and by then eliminating δv^a itself in a way that we now describe.

We focus on the normal-mode solutions to the above perturbation equations, i.e. those solutions with time dependence $e^{i\omega t}$ and azimuthal-angle dependence $e^{im\phi}$, where ω is the mode frequency and m is an integer. For these modes, (2.54) yields an algebraic expression for ξ^a in terms of δv^b (or vice versa):

$$\xi_a = -i(g_{ab}/\sigma - (i\phi_a \nabla_b \Omega)/\sigma^2)\,\delta v^b, \tag{2.55}$$

where $\sigma \equiv \omega + m\Omega$. Equation (2.53) now takes the form

$$\delta\rho = (\rho/\gamma p)\,\delta p + (i\rho^2/\sigma)\,A_a\,\delta v^a, \tag{2.56}$$

where
$$A_a \equiv (\nabla_a \rho)/\rho^2 - (\nabla_a p)/\gamma\rho p. \tag{2.57}$$

Note that $A_a = 0$ for barotropic configurations, which have adiabatic index $\gamma = (\rho/p)(dp/d\rho)$. For general configurations, such as those in which the pressure is not a unique function of the density, A_a does not vanish. With $\delta\rho$ eliminated via (2.56), (2.53) has the representation

$$i\sigma Q^{-1}_{ab}\,\delta v^b \equiv (i\sigma g_{ab} + 2\nabla_b v_a - \phi_a \nabla_b \Omega - (i/\sigma)\nabla_a p A_b)\,\delta v^b = -\nabla_a \delta U - \rho(\delta U + \delta\Phi)\,A_a, \tag{2.58}$$

where
$$\delta U \equiv (\delta p)/\rho - \delta\Phi. \tag{2.59}$$

Notice that (2.58) is algebraic in δv^a and can be solved for δv^a as long as Q^{-1}_{ab} is invertible. In this case the solution to (2.58) for δv^a is

$$\delta v^a = iQ^{ab}\nabla_b \delta U + i\rho(\delta U + \delta\Phi)\,Q^{ab}A_b, \tag{2.60}$$

where

$$Q^{ab} = (\lambda/\sigma^3)[(\sigma^2 - A^c \nabla_c p)\,g^{ab} - 2\omega^a\Omega^b + i\sigma\phi^a\nabla^b\Omega - 2i\sigma\Omega\,\nabla^a\phi^b + \nabla^a p A^b + (i/\sigma)\,\phi^a\phi^{bc}\nabla_c p A^d\omega_d - (2i/\sigma)\,\phi^b\phi^{ac}A_c\Omega^d\nabla_d p]. \tag{2.61}$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Here $\quad \det Q^{-1a}{}_b \equiv \sigma^3/\lambda = (1/\sigma)(\sigma^4 - \sigma^2 A^a \nabla_a p - 2\sigma^2 \Omega^a \omega_a + 2A^a \omega_a \Omega^b \nabla_b p);$ (2.62)

$\omega^a = \epsilon^{abc}\nabla_b v_c$ and is the fluid vorticity; ϵ_{abc} is the antisymmetric tensor (components $= \pm 1$ in cartesian coordinates); $\Omega^a = \Omega z^a$, where z^a is a unit vector parallel to the rotation axis, and is the fluid angular-velocity vector; and $\phi^{ab} = \epsilon^{abc}\phi_c/\phi^d\phi_d$.

We use (2.60) to eliminate δv^a in favour of δU and $\delta \Phi$. And we eliminate $\delta\rho$ by combining (2.50), (2.59), and (2.56), which yield

$$\delta\rho = \Psi_1(\delta U + \delta\Phi) - (\rho^2/\sigma) A_a Q^{ab} \nabla_b \delta U, \tag{2.63}$$

where
$$\Psi_1 = \rho^2/\gamma p - (\rho^3/\sigma) A_a Q^{ab} A_b. \tag{2.64}$$

We have not yet used (2.50) and (2.52). It is these that now provide the fundamental set of coupled eigenequations for the two potentials δU and $\delta\Phi$. Using (2.59), (2.60), and (2.63) to eliminate δp, $\delta\rho$, and δv^a, we are able to re-express (2.50) and (2.52) as the fourth-order system

$$\nabla_a(\rho Q^{ab}\nabla_b \delta U) + \Psi_3 \delta U = -\rho^2 Q^{ab} A_b \nabla_a \delta\Phi - \Psi_2 \delta\Phi, \tag{2.65}$$

$$\nabla^a\nabla_a \delta\Phi + 4\pi G \Psi_1 \delta\Phi = (4\pi G\rho^2/\sigma) A_a Q^{ab}\nabla_b \delta U - 4\pi G \Psi_1 \delta U, \tag{2.66}$$

where
$$\Psi_2 = \sigma\Psi_1 + \nabla_a(\rho^2 Q^{ab} A_b), \tag{2.67}$$

$$\Psi_3 = \Psi_2 - (m\lambda/\sigma^3)\rho^2(\sigma A^a \nabla_a \Omega + (4\sigma\Omega/\varpi) A^a \nabla_a \varpi + (1/\sigma) A_a \phi^{ab}\nabla_b p A^c \omega_c). \tag{2.68}$$

It is easy to verify that (2.65) and (2.66) are real equations for $\delta U(z,\varpi)$ and $\delta\Phi(z,\varpi)$, where $\delta U = \delta U(z,\varpi)\,\mathrm{e}^{\mathrm{i}\omega t + \mathrm{i}m\varphi}$ and $\delta\Phi = \delta\Phi(z,\varpi)\,\mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}m\varphi}$.

We complete the specification of the eigenvalue problem by imposing appropriate boundary conditions at the stellar surface and at $r = \infty$, where r is the spherical radial coordinate. At the stellar surface the appropriate boundary condition is that the lagrangian change in the pressure vanishes, i.e. that

$$\nabla p = \delta p + \xi^a \nabla_a p = 0 \tag{2.69}$$

at the stellar surface. It follows from (2.55), (2.59), and (2.60) that this condition can be rewritten as

$$\delta U + \delta\Phi + (\nabla_a p/\rho) Q^{ab}[\nabla_b \delta U + \rho A_b(\delta U + \delta\Phi)] = 0. \tag{2.70}$$

The remaining boundary condition is that $\delta\Phi \rightarrow 0$ sufficiently rapidly as $r\rightarrow\infty$. For a perturbation with φ-coordinate dependence $\mathrm{e}^{\mathrm{i}m\varphi}$, examination of the expansion of $\delta\Phi$ as a power series in negative powers of r reveals that this condition can be expressed as

$$\frac{\partial}{\partial r}\delta\Phi + \sum_{l\geqslant|m|}\frac{(l+1)(2l+1)(l-m)!}{2r(l+m)!}P_l^m(\mu)\int_{-1}^{1}\delta\Phi(r,\mu')P_l^m(\mu')\,\mathrm{d}\mu' = 0, \tag{2.71}$$

where $\mu = \cos\theta$, θ is the polar angular coordinate, and P_l^m is the associated Legendre function.

(iii) *The numerical method of solution*

We next describe briefly a numerical method that has been developed and used to solve the eigenequations directly for ω, δU, and $\delta\Phi$. For more details, the reader is referred to the extensive discussion presented elsewhere (Ipser & Lindblom 1990).

In the method of Ipser & Lindblom, the equations describing the equilibrium of a stellar model and the eigenequations (2.65), (2.66), (2.70), (2.71) describing a normal mode of pulsation are first written out in spherical coordinates r, $\mu = \cos\theta, \varphi$, and are then represented as difference equations on a finite grid. The chosen grid consists of uniformly spaced points along $(2L-1)$ radial spokes that emanate from the origin at the zeros μ_i the Legendre polynomial P_{2L-1}. This choice of the angular location of spokes permits one to accurately represent integrals of a function over μ, and its various derivatives with respect to μ, as weighted averages of the values of the function at the various μ_i. Radial derivatives of a function are represented in terms of standard three-point difference formulae. For convenience, the boundary conditions (2.70) is imposed not only at the last point inside the star along each spoke but also at all points outside the star out to the last grid point along the spoke. This amounts to a smooth extension of the definition of δU into the exterior region, and it simplifies the numerical analysis. The boundary condition (2.71) is imposed at the last grid point along each spoke.

When this procedure is completed, the eigensystem takes the form of a set of coupled linear equations for the values of δU and $\delta \Phi$ at the grid points:

$$\sum_b B_a{}^b\, \delta U_b = \sum_b C_a{}^b\, \delta\phi_b, \tag{2.72}$$

$$\sum_b D_a{}^b\, \delta \Phi_b = \sum_b E_a{}^b\, \delta U_b, \tag{2.73}$$

where the indices label the grid points. Equations (2.72) and (2.73) can be combined to yield a single equation for either δU_a or $\delta \Phi_a$, but it turns out to be more efficient to leave them as separate equations and to solve them iteratively as follows. Given an estimate $\delta U^{(i)}$, $\delta \Phi^{(i)}$, $\omega^{(i)}$ for the eigenfunctions and eigenvalue (taken, perhaps, from the solution obtained for a slightly different stellar model), (2.73) is solved for the new estimate $\delta \Phi^{(i+1)}$, which is then used in (2.72) to find $\delta U^{(i+1)}$. Next, a suitable average $\langle \delta U \rangle$ of the values δU_a is used to compute the quantity $z^{(i)} \equiv \langle \delta U^{(i+1)} \rangle / \langle \delta U^{(i)} \rangle$, which monitors the average change of δU from one iteration to the next. The quantity $z^{(i)}$ is used to update the eigenvalue in a way that attempts to predict the value that will make $z^{(i+1)}$ as close to unity as possible:

$$\omega^{(i+1)} = \omega^{(i)} + s(1-z^{(i)})\,(\omega^{(i)} - \omega^{(i-1)})/(z^{(i)} - z^{(i-1)}).$$

Here $s \leqslant 1$ is a convergence factor chosen to maximize stability of the iteration process. Iteration continues until convergence is achieved.

(iv) *Results*

We now describe very briefly the results obtained by applying the above method to study the $l = m$ f-modes, those having no radial nodes in the non-rotating limit of uniformly rotating polytropes (see Ipser & Lindblom (1990) for details). These configurations have an equation of state and adiabatic index given by

$$p = \kappa \rho^{1+1/n}, \quad \gamma = 1 + 1/n, \tag{2.74}$$

which implies $A_a = 0$.

The dependence of the eigenfrequency ω on the angular velocity Ω is conveniently expressed in terms of the function

$$\alpha_m(\Omega) = (\omega(\Omega) + m\Omega)/\omega(0). \tag{2.75}$$

Figs 1–6 of Ipser & Lindblom (1990) reveal that $\alpha_m(\Omega)$ is a slowly varying function of Ω and of the polytropic index n. Its value generally varies by less than 30% from unity in uniformly rotating models. For given n, $\alpha_m(\Omega)$ typically rises from unity at $\Omega = 0$ to a maximum value $\lesssim 1.2$ and then drops to a value somewhat less than unity (by 20% or so) as Ω_a reaches the value Ω at which sequences terminate due to equatorial shedding. The eigenfrequency ω passes through zero at a value Ω_{crit} whose ratio $\Omega_{crit}/\Omega_{max}$ increases with decreasing m and with increasing n. This ratio has the value 0.878 for $m = 4$ and $n = 1$, for example. Figs 7–14 of Ipser & Lindblom (1990) reveal that the eigenfunctions δU of modes become more highly peaked near the stellar surface as Ω increases, as n increases, and as the angular indices $l = m$ increase. For a typical mode, the solution for δU yields lagrangian displacement vectors with nearly equal radial and axial components that are out of phase. This indicates that the motion of a fluid particle is approximately circular motion about its equilibrium position.

In the absence of viscous dissipation, gravitational radiation reaction induces a secular instability in the $l = m$ f-modes at the point where ω passes through zero. Now that the exact eigenfunctions can be obtained by the above method, one can provide a much more reliable estimate, for a given viscosity model, of viscous effects on the instability. As is discussed elsewhere in this paper (also Ipser & Lindblom 1989, 1991*a*), analyses using the exact eigenfunctions indicate that viscous effects stabilize all modes to the secular instability unless $\Omega/\Omega_{max} \gtrsim 0.9$. Using a slow-rotation approximation, Weber & Glendenning (1991) report a substantially lower estimate of the instability point. We believe that the Ipser–Lindblom estimate is more accurate, but a precise determination of the general-relativistic instability points is overdue.

(*c*) *Stability*

Two kinds of instability limit the region of uniformly rotating relativistic stars. An axisymmetric instability sets upper and lower limits on the central density and baryon number at fixed angular momentum; and a non-axisymmetric instability, driven by gravitational radiation can set a limit more stringent than the Kepler frequency on the maximum angular velocity.

(i) *Axisymmetric instability*

For spherical stars, any perturbation can be written as a superposition of spherical harmonics that are axisymmetric about some axis, and one therefore need only consider stability of axisymmetric perturbations. In fact, apart from local instability to convection when the temperature gradient becomes super-adiabatic, one need only consider radial perturbations (Detweiler & Ipser 1973). For radial oscillations of the Schwarzschild geometry, the functional E was first obtained by Chandrasekhar (1964). In newtonian gravity, instability sets in when the matter becomes relativistic, when the adiabatic index γ (more precisely, its pressure-weighted average) reaches the value $\frac{4}{3}$ characteristic of zero rest mass particles. In relativity, even models with the stiffest equation of state must be unstable to collapse for some value of $R/M > \frac{9}{8}$, the ratio for the most relativistic model of uniform density. For stars with γ near $\frac{4}{3}$, general relativity may be important for radii large compared with M (Fowler 1964; Chandrasekhar 1964). To determine the instability points is simple in practice if one models neutron stars by one-parameter EOS. Misner & Zapolsky (1964) noticed that, along a sequence of such neutron star models, the configuration at which the functional E first becomes negative appeared to be the model with maximum mass. The reason is that by using a one-parameter state EOS, one makes the approximation that the adiabatic index governing the pulsations is the polytropic index of the equilibrium star, that

$$\gamma = \frac{\mathrm{d}p/\mathrm{d}r}{p}\frac{\epsilon+p}{\mathrm{d}\epsilon/\mathrm{d}r}. \tag{2.76}$$

Then a turning-point method, due initially to Poincaré, implies that points at which the stability of a mode changes are extrema of the mass (Oppenheimer & Volkoff 1939; Harrison *et al.* 1958; see Thorne (1967, 1978) for later references and a review of the turning point method applied to spherical neutron stars; a somewhat different treatment is given by Zel'dovich & Novikov (1971)).

For dynamical oscillations of neutron stars the adiabatic index does not coincide with the polytropic index of the star, and the turning point method locates a secular instability, one whose growth time is long compared with the typical dynamical time of stellar oscillations. The turning-point instability proceeds on a timescale slow enough to accommodate the nuclear reactions and energy transfer that accompany the change to a nearby equilibrium. In the case of rotating stars, the turning point similarly marks a zero-frequency 'mode', in the sense of a time-independent perturbation of the star. The perturbation is again not dynamical, and in this case the timescale must also be long enough to accommodate the transfer of angular momentum needed to keep the angular velocity uniform. That is, the growth rate of the instability is limited by the time required for viscosity to redistribute the star's angular momentum. For neutron stars, this is expected to be short, probably comparable with the spin-up time following a glitch, and certainly short compared with the lifetime of a pulsar or an accreting neutron star.

The argument can be stated heuristically as follows. When the mass has a maximum along a curve of constant J, the total baryon number turns over as well, because of the relation (Bardeen 1972)

$$\mathrm{d}M = \Omega\,\mathrm{d}J + \mu\,\mathrm{d}N. \tag{2.77}$$

At the turning point, nearby models have (to first order in the path parameter ϵ) the same angular momentum, baryon number and mass. The corresponding perturbation relating two such equilibria is then a time-independent solution to the linearized equations of a perfect fluid in general relativity, but a solution for which the angular momentum of each fluid element changes (when the star is rotating).

Models on the high density side of the instability point are unstable because the injection energy is a decreasing function of central density. The relation can be understood if one considers again a sequence of stars with fixed angular momenta. The turning point is a star with maximum mass and baryon number, and on opposite sides of the turning point are corresponding models with the same baryon number. Because $\mu = \partial M/\partial N$ is a decreasing function of central density, the model on the high-density side of the turning point has a greater mass than the corresponding model with smaller central density. For spherical neutron stars, the low-density endpoint of the equilibrium sequence is again an extremum of the mass, in this case a minimum. For rapidly rotating stars, however, only the high density endpoint of a constant J sequence is an extremum of the mass. As the density is lowered at fixed J, the binding energy decreases and the sequence terminates by mass shedding: the equator rotates with angular velocity equal to that of a particle in keplerian orbit.

The stability criterion in the form discussed above is stated as the theorem below. An important feature of the criterion, however, is its independence of the parameters chosen to label the equilibrium models. We therefore first state the principal result in a way that does not restrict the parametrization. A stellar model will mean a stationary, axisymmetric solution to the Einstein field equations with perfect fluid source.

Lemma. *Consider a two-parameter family of uniformly rotating stellar models having an* EOS *of the form* $p = p(\epsilon)$. *Suppose that along a continuous sequence of models labelled by a parameter* λ, *there is a point* λ_0 *at which both* $\dot{N} (= \mathrm{d}N/\mathrm{d}\lambda)$ *and* $\dot{J}$ *vanish and where* $(\dot{\Omega}\dot{J} + \dot{\mu}\dot{N})^{\cdot} \neq 0$. *Then the part of the sequence for which* $(\dot{\Omega}\dot{J} + \dot{\mu}\dot{N}) > 0$ *is unstable for* λ *near* λ_0.

Proof. The result follows from Theorem I of Sorkin (1982), with his function S replaced by $-M$, with E^{α} replaced by N and J, and with his β_{α} replaced by μ and Ω. The conditions of the theorem are satisfied because stellar models are configurations for which M is minimized at fixed N and J.

One may phrase the criterion in terms of an extremum of the mass or rest mass, choosing a sequence of stars along which the total angular momentum is constant. Alternatively, one may use an extremum of the angular momentum, choosing a sequence of stars along which the total rest mass is constant. This latter form is emphasized by Cook *et al.* (1992).

Theorem (Friedman *et al.* 1988). *Consider a continuous sequence of uniformly rotating stellar models based on an* EOS *of the form* $p = p(\epsilon)$. *Suppose that the total angular momentum is constant along the sequence, and that there is a point* λ *where* $\dot{M} = \mathrm{d}M/\mathrm{d}\lambda = 0$ (*and where* $\mu > 0$, $(\dot{\mu}\dot{M})^{\cdot} \neq 0$). *Then the part of the sequence for which* $\dot{\mu}\dot{M} > 0$ *is unstable for* λ *near* λ_0.

The turning point is similarly an extremum of J along a sequence of constant N. Note that the theorem is a straightforward application of the turning point method in the form given by Sorkin (1981, 1982), and it requires no assumptions about existence or completeness of normal modes.

Another way to describe the instability should be mentioned. The stellar models comprise a two-parameter set, Σ, the surface of figure 1. The map that assigns to each star its mass M and angular momentum J is a projection of Σ onto the M–J plane. In most places the map is 1–1, but along a curve l it becomes non-invertible, because the sheet doubles back on itself, leaving a portion of the M–J plane uncovered. On one side of the image of l in the M–J plane lie values of mass and angular momentum to which no model corresponds. On the other side, in a neighbourhood of l, there are two distinct stars belonging to separate branches of Σ. Of these two stars, one might expect the one with greater energy to be unstable, because, by definition, there is another configuration with lower energy. The above theorem justifies this expectation.

For a star that becomes unstable, the outcome of the axisymmetric instability depends on whether the mass is a maximum or a minimum. Matter accreting on a neutron star that is at its maximum mass will lead to gravitational collapse. If, on the other hand, a neutron star is near its lower mass limit, loss of matter (for example, by tidal stripping in the coalescence of a binary system of two neutron stars) leads to explosion, because the unstable models have positive binding energy, they are unbound.

To locate points of dynamical stability to axisymmetric perturbations along sequences of rotating relativistic stars, one must evaluate the functional E of (2.41)

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
THE ROYAL SOCIETY A
PHILOSOPHICAL TRANSACTIONS OF

for a set of trial functions (Schutz 1972; Chandrasekhar & Friedman 1972*b*), and no explicit calculations have as yet been attempted. Fortunately, in addition to its simplicity, the turning-point criterion has the virtue that secular rather than dynamical instability can be expected to limit the class of observed neutron stars: a sequence of uniformly rotating neutron stars becomes dynamically unstable after it has become secularly unstable.

(ii) *Non-axisymmetric instability*

For hot neutron stars, non-axisymmetric instability appears to set an upper limit on rotation more stringent than Ω_K. The instability is driven by gravitational waves, and arises in the following way. For slowly rotating stars, gravitational waves carry positive angular momentum, J, from a forward mode and negative J from a backward mode, thus damping all non-axisymmetric perturbations.

But when $m\Omega \sim \sigma$, the mode that moves backward relative to the star is dragged forward relative to an inertial observer. Gravitational waves now carry positive angular momentum from a mode that already has $J<0$ (the perturbed star has lower J because the perturbation moves backward relative to the star) so gravitational radiation drives the mode. The Dedekind bar instability found by Chandrasekhar (1970) is the $m=2$ case of this mechanism, but higher modes are unstable first, and neutron stars probably reach the Kepler frequency before an $m=2$ mode can ever become unstable.

When the growth time of a mode is longer than the viscous damping time, viscosity will stabilize the mode. Viscosity rises sharply as a neutron star cools, and only newly formed or, possibly, accreting neutron stars are hot enough for the non-axisymmetric instability to play a role.

Formally, in the absence of viscosity, the instability follows from the form of the second order change E in the energy, for a canonical displacement ξ^a. That is, if E is negative for any initial data $(\xi, \mathcal{L}_t \xi, h_{ab}, \mathcal{L}_t h_{ab})$ on a spacelike hypersurface S, satisfying the constraint equations

$$\delta(G^{ab} - 8\pi T^{ab})\, n_b = 0, \tag{2.78}$$

and preserving vorticity (equation (2.44)), then the configuration is unstable or marginally unstable: there exist perturbations which do not die away in time. If E is strictly less than zero, the time-derivatives $\mathcal{L}_t \xi$ and $\mathcal{L}_t h_{ab}$ remain finitely large for all times, and the configuration will be strictly unstable unless there are time-dependent solutions to the perturbation equations which do not radiate For non-axisymmetric perturbations, this seems clearly impossible, but there is no formal proof.

In particular, by a choice of ξ^a orthogonal to the Killing vectors d and ϕ, the energy functional has the form

$$E = -\tfrac{1}{2}(m\Omega)^2 \int (\epsilon + p)\, \xi^2\, \mathrm{d}V + O(m), \tag{2.79}$$

negative for sufficiently large m. Explicit points of instability along the Maclaurin sequence of uniformly rotating, uniform-density newtonian models were found by Comins (1979*a*, *b*) for the $l=m$ modes (for each value of m these become unstable at the lowest value of Ω) and by Baumgart & Friedman (1986) for a larger set of modes. For polytropic newtonian models with stiffness comparable to that of neutron stars, instability points of modes with $m = 3$, 4 and 5 were found by Imamura *et al.* (1985)

and Managan (1985). Managan used the eulerian variational principle mentioned in §1, obtained by algebraically solving Euler's equations for the change $\delta \boldsymbol{v}$ in the fluid velocity. These results were subsequently confirmed by the explicit construction of normal modes (Ipser & Lindblom 1990).

For a perfect fluid, the growth time of an unstable mode (and the damping time of a stable mode) is the radiation reaction time. For perturbations of spherical stars, with angular dependence $Y_{lm}(\theta, \phi)$, this has the approximate value (Detweiler 1975)

$$\tau \approx \frac{(l-1)[(2l+1)!!]^2}{(l+1)(l+2)}\left[\frac{2l+1}{2l(l-1)}\right]^l \left(\frac{R}{M}\right)^{l+\frac{1}{2}} \tau_{\mathrm{dyn}}, \tag{2.80}$$

rapidly increasing as l increases. Work by Detweiler & Lindblom (1977) suggested that viscosity would stabilize any mode whose growth time was longer than the viscous damping time, and this was confirmed by Lindblom & Hiscock (1983).

For an imperfect fluid, the lagrangian change $\Delta_{\mathrm{PF}} T^{ab}$ in the perfect fluid stress tensor (equation (2.35)) is modified by the addition of terms involving viscosity and thermal conductivity. If η and ζ are the coefficients of shear and bulk viscosity, and $\delta\sigma^{ab}$, $\delta\theta$, and δq^a are the changes in shear, expansion and heat flow (assumed to vanish for the equilibrium model), one has (Lindblom & Hiscock 1983)

$$\Delta T^{ab} = \Delta_{\mathrm{PF}} T^{ab} + (nTu^a u^b + (\partial p/\partial s)\, q^{ab})\, \delta s - 2\eta\, \delta\sigma^{ab} - \zeta q^{ab}\, \delta\theta + 2u^{(a}\delta q^{b)}. \tag{2.81}$$

The corresponding change in the Euler equations is given by

$$0 = \Delta \nabla_b T^{ab} = \Delta_{\mathrm{PF}} \nabla_b T^{ab} - F^b, \tag{2.82}$$

where
$$F^b = \nabla_b\{2\eta\, \delta\sigma^{ab} + \zeta q^{ab}\, \delta\theta - 2u^{(a}\delta q^{b)} - (nTu^a u^b + (\partial \rho/\partial s)\, q^{ab})\, \Delta s\}, \tag{2.83}$$

and the loss in the energy of a perturbation from one hypersurface to another has the form

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)_{\mathrm{GRR}} - \int \mathrm{d}\tau \frac{1}{u^t}\left\{2\eta\, \delta\sigma^{ab}\, \delta\sigma_{ab} + \zeta(\delta\theta)^2 + \frac{1}{kT} q^{ab}\, \delta q_a\, \delta q_b\right\}. \tag{2.84}$$

Detailed computations of the viscous damping time have been carried out by Cutler *et al.* (1990) for normal modes of spherical stars. They use a shear viscosity dominated by electron scattering in the superfluid with η given in c.g.s. units by (Flowers & Itoh 1976)

$$\eta = 6.0 \times 10^{18} (\rho_{15}/T_{\mathrm{g}})^2.$$

When temperatures are high enough to preclude superfluidity, neutron scattering dominates the kinematical viscosity. The bulk viscosity for normal neutron matter has been estimated by Sawyer (1989), who obtains (in c.g.s. units)

$$\zeta_n = 6.0 \times 10^{25} (\rho_{15}/\sigma)^2\, \mathrm{e}^{\nu} (T_{\mathrm{g}})^6.$$

(Sawyer also gives estimates for pion condensates and strange quark matter.) For temperatures below about $5 \times 10^9 K$, shear viscosity dominates.

However, at temperatures below the superfluid transition temperature, Lindblom & Mendell (1992) claim that friction between electrons and neutron vortices is the dominant dissipative mechanism and that it is always large enough to damp the non-axisymmetric instability.

This means that there is only a narrow window of temperatures for which the non-axisymmetric instability limits the rotation of neutron stars: It appears that above $2 \times 10^{10} K$, bulk viscosity damps all modes and below about $10^9 K$, the Lindblom–

Mendell mechanism damps all modes. Neutron stars formed from the collapse of accreting white dwarfs are likely to have their rotation limited by the instability. If Lindblom & Mendell are correct, however, neutron stars spun up by accretion will never be hot enough to be unstable. The class of accretion-driven neutron stars may therefore have a slightly higher limiting frequency than the class of neutron stars with dwarf progenitors.

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